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A Cosmological exact solution of generalized Brans-Dicke theory with complex scalar field and its phenomenological implications

Research Article

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Abstract: When the Brans-Dicke theory is formulated in terms of the Jordan scalar field φ, the amount of dark energy is related to the mass of this field. We investigate a solution which is relevant to the late universe. We show that if φ is taken to be a complex scalar field, then an exact solution to the vacuum equations requires that the Friedmann equation possesses both a constant term and one which is proportional to the inverse sixth power of the scale factor. Possible interpretations and phenomenological implications of this result are discussed.

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1. Introduction

The simplest and earliest scalar-tensor theory [1, 2] considered a massless scalar field φ that couples to the Ricci Scalar; this scalar field replaces the gravitational constant in the Einstein-Hilbert action by a term proportional to φ^2. Brans and Dicke redefined the scalar field so that $\Box\phi = 0$ is a vacuum solution [3]. In the Jordan-Brans-Dicke theories (JBD), the counterpart of the gravitational coupling term, $1/16\pi G_N$, is replaced by ϕ or $\varphi^2/8\omega$, which may be a function of space and/or time. For an isotropic homogeneous cosmology, which evolves in time, the scalar field is a function only of time. JBD gravity has been used extensively to develop dark energy models [4–8, 10], which usually involve a scalar potential which is adjusted to fit the observed cosmology. In contrast, we use a simple model where the scalar field is complex: the complex Jordan-Brans-Dicke (CJBD) model. Besides the kinetic term, Lagrangian density contains only a standard mass term for the Jordan field φ, so that our Lagrangian density in flat

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spacetime reduces to the standard Lagrangian density of a free complex massive scalar field. This is in contrast to most cosmological models based on Jordan-Brans-Dicke theory where the potential $V(\varphi)$ is adjusted to give the observed cosmology. Making the scalar field complex may account for an additional energy density which, through the field equation, may mimic the expansion usually attributed to dark matter. A complex scalar field was used as the source of accelerated expansion in [11]. In [12], a complex scalar field that does not couple to the Ricci scalar was used with its phase acting as quintessence and its amplitude causing primordial inflation. In a different scenarion, Briscese [13] also used a self-interacting complex scalar field that does not couple to the Ricci scalar. In their scenario, the inflation field decays into a scalar field, and a Bose-Einstein condensate of scalar field particles behaves as dark matter.

Addition of phase to a scalar field is similar to two real fields with the same mass. Two real scalar fields with different masses have also been considered in cosmology [14, 15]. In view of these studies, the idea to merge a complex scalar field with JBD theory appeared interesting. We will show that this merge results in a term proportional to $\frac{1}{a^6}$ in the Friedmann equation. The CP (charge-parity) - violating term in QCD (Quantum chromodynamics) has a coupling which is a pure phase. A model which turns this phase into a particle results in a particle called an axion [16, 17], a dark matter candidates. The fact that the CP-violating parameter in QCD is a phase is also motivation for considering a phase for the JBD scalar field. In [18], a complex bulk scalar field was considered to be dark matter in the Randall-Sundrum Model, and [19] used a complex scalar field as a unified source of dark energy and dark matter of the Universe.

In Section 2, an exact cosmological solution is displayed, with a stability analysis for the vacuum case included in the Appendix. The effect of the proposed complex scalar field on the evolution of the late-time Universe is obtained in Section 3. We show that while the mass of the CJBD scalar contributes to the constant term in the Friedmann equation interpreted as dark energy, its *phase* contributes a term proportional to the inverse sixth power of the scale factor. A complex scalar field can be considered as two real fields such that the Lagrangian density is invariant under $SO(2) \equiv U(1)$.

$$\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} \tag{1}$$

$$\varphi = \varphi_1 + i\varphi_2 \to e^{i\alpha}(\varphi_1 + i\varphi_2) \tag{2}$$

The cosmological solution presented in this paper is invariant under constant translations of the phase $\chi \to \chi + \chi_0$. We use a metric signature $(+ - --)$ and the Jordan formalism. The Lagrangian density of JBD theory is given as

$$\pounds_{JBD} = -\frac{\varphi^2}{8\omega}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi. \tag{3}$$

It is possible to transform a Lagrangian density of the form $\pounds_{BD} = \frac{\varphi^2}{8\omega}R + ...$ into Einstein frame $\pounds_{EH} = \frac{1}{8\pi G_N}R + ...$ by considering a transformation which mixes the scalar field and metric. Different physical cosmologies are present in the two frames, meaning that only one of the frames can be chosen as physical; we chose the Jordan frame. The reason of this choice that, for flat space-time, $\pounds_{JBD}$ should reduce to the standard Lagrangian density of a complex massive scalar (Klein-Gordon) field. We add a mass term and a phase χ,

$$\varphi = \varphi_1 + i\varphi_2 = |\varphi|\, e^{i\chi}, \tag{4}$$

so that the Lagrangian density with a complex massive scalar field becomes:

$$\pounds = -\frac{\varphi\varphi^*}{8\omega}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi^* - \frac{1}{2}m^2\varphi\varphi^*, \tag{5}$$

The action is defined by

$$S = \int d^4x\sqrt{-g}\pounds + S_M. \tag{6}$$

When this action is varied with respect to the metric and the complex scalar field, the equations of motion, for a Friedmann-Robertson-Walker metric and perfect fluid energy-momentum tensor $T^\mu_\nu = diag\,(\rho, -p, -p, -p)$, reduce to the following:

$$\frac{3}{4\omega}\varphi\varphi^*\left(\frac{\dot a^2}{a^2}+\frac{k}{a^2}\right) - \frac{1}{2}\dot\varphi\dot\varphi^* - \frac{1}{2}m^2\varphi\varphi^* + \frac{3}{4\omega}\frac{\dot a}{a}(\dot\varphi\varphi^* + \varphi\dot\varphi^*) = \rho_M, \tag{7}$$

$$\frac{-1}{4\omega}\varphi\varphi^*\left(2\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+\frac{k}{a^2}\right) - \frac{1}{2\omega}\frac{\dot a}{a}(\dot\varphi\varphi^*+\varphi\dot\varphi^*) - \frac{1}{4\omega}(\ddot\varphi\varphi^*+\varphi\ddot\varphi^*) - \left(\frac{1}{2}+\frac{1}{2\omega}\right)\dot\varphi\dot\varphi^* + \frac{1}{2}m^2\varphi\varphi^* = p_M, \tag{8}$$

$$\ddot\varphi + 3\frac{\dot a}{a}\dot\varphi + \left[m^2 - \frac{3}{2\omega}\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+\frac{k}{a^2}\right)\right]\varphi = 0. \tag{9}$$

Equation (9), being complex, is equivalent to two real equations. H and $F_1 + iF_2$ are respectively defined as the fractional rate of changes of the scale size and the JBD scalar field φ.

$$H = \frac{\dot{a}}{a}, \quad F_1 + iF_2 = \frac{\dot{\varphi}}{\varphi}, \tag{10}$$

where F_1 and F_2 are real, and defined as

$$F_1 = |\varphi|^{-1}\frac{d\,|\varphi|}{dt} = \frac{\dot{\varphi}_1\varphi_1 + \dot{\varphi}_2\varphi_2}{\varphi_1^2 + \varphi_2^2} = -\frac{1}{2}\frac{\dot{G}_N}{G_N}, \tag{11}$$

$$F_2 = \frac{\dot{\varphi}_2\varphi_1 - \dot{\varphi}_1\varphi_2}{\varphi_1^2 + \varphi_2^2} = \dot{\chi}. \tag{12}$$

For a spatially flat ($k = 0$) universe, we will obtain solutions which give H, F_1 and F_2 as a function of the scale size a. After rewriting equations (7), (8), and (9) in terms of H, F_1, F_2 and their derivatives with respect to the scale size of the universe, a, we get the following equations:

$$3H^2 - 2\omega(F_1^2 + F_2^2) + 6HF_1 - 2\omega m^2 = \frac{4\omega\rho_M}{|\varphi|^2} \tag{13}$$

$$3H^2 + (2\omega + 4)F_1^2 + 2\omega F_2^2 + 4HF_1 + 2aH(F_1' + H') - 2\omega m^2 = \frac{-4\omega p_M}{|\varphi|^2} \tag{14}$$

$$-6H^2 + 2\omega F_1^2 - 2\omega F_2^2 + 6\omega HF_1 + 2a\omega HF_1' - 3aHH' + 2\omega m^2 = 0 \tag{15}$$

$$(4\omega F_1 + 6\omega H)F_2 + 2\omega aHF_2' = 0 \tag{16}$$

where primes denotes derivatives with respect to a.

2. The exact cosmological solutions for the dust-dominated case and the vacuum case

Exact cosmological solutions of the JBD theory can be obtained by analyzing the symmetries of the field equations [20]. Perturbative solutions were found in [21, 22]. We propose an ansatz for finding the exact solutions for the late-time universe, and show that an exact solution of the cosmological equations can be obtained by an ansatz for both the matter dominated case and the vacuum case.

$$F_1(a) = \frac{H(a)}{2(\omega + 1)} \tag{17}$$

$$\frac{|\varphi|'}{|\varphi|} = \frac{F_1}{aH}, \tag{18}$$

where prime denotes derivative with respect to a. By integrating equation (18) with the ansatz yields

$$|\varphi^2| = |\varphi_0^2|\left[\frac{a}{a_0}\right]^{\frac{1}{(1+\omega)}}. \tag{19}$$

Since $\omega > 10^4$ [23–26] $|\varphi|$ varies very slowly as the universe evolves, so φ is approximately constant during the matter dominated era. The massive JBD theory, where the scalar field, ϕ in Eq. (5) is real, was also analysed in [27, 28]. For the massive JBD theory, the PPN (Parameterized post-Newtonian formalism) parameter γ is given by

$$\gamma = \frac{3 + 2\omega - e^{-m_0 r}}{3 + 2\omega + e^{-m_0 r}}. \tag{20}$$

If the scalar field is light, there is no difference between massive and massless case on the PPN parameters and

$$\gamma = \frac{1 + \omega}{2 + \omega}. \tag{21}$$

When the phase is taken as constant for the Schwarzschild solution, the complex scalar field turns into the amplitude, becoming equivalent to a real field, and does not change the PPN parameters. Variation of a gravitational constant which is proportional to F_1 in equation (11) is also small due to its proportionality to $\frac{1}{\omega}$.

By using the ansatz, equation (17), equation (16) gives the solution

$$F_2 = F_{20}\left[\frac{a}{a_0}\right]^{\alpha} \tag{22}$$

$$\alpha = -\left(3 + \frac{1}{(1+\omega)}\right) \tag{23}$$

since $\omega > 10^4$ for all practical purposes $\alpha = -3$. For the dust solution of the complex JBD model, $\rho_M = \rho_0\left(\frac{a}{a_o}\right)^{-3}$ and $p_M = 0$; when equation (17) and (22) are placed into equation (13), equations (14), (15) are satisfied and the H^2 relation is derived as

$$H^2 = \frac{4\omega(1+\omega)^2}{(3\omega+4)(2\omega+3)} \left[m^2 + \frac{2\rho_0}{|\varphi_0|^2}\left(\frac{a}{a_0}\right)^{\alpha} + F_{20}^2\left(\frac{a}{a_0}\right)^{2\alpha}\right]. \tag{24}$$

Several interesting features emerge for the vacuum case, $\rho_M = p_M = 0$. Then H^2 contains only a constant term and a $1/a^6$ term, respectively. Since F_2 is directly related to the phase of the scalar field in equation (12), the $1/a^6$ term is identified with the effect of the phase on the expansion of the universe,

$$H^2(a) = \frac{4\omega(1+\omega)^2}{(3\omega+4)(2\omega+3)}(m^2 + F_2^2(a)). \quad (25)$$

When we restrict the model to the $F_2 = 0$ case, $F = F_1$, the complex scalar field turns into a real scalar field and previously studied JBD equations are obtained [22]. Appendix details an investigation of the stability of this proposed solution against pertubations.
With the addition of the cosmological constant to the cold dark matter (CDM) model, the resulting ΛCDM model gives a better fit [29, 30]. Five-year Wilkinson Microwave Anisotropy Probe (WMAP) temperature and polarization observations [31] which included data from Baryon Acoustic Oscillations in the Galaxy and Type Ia supernova luminosity/time dilation measurements [33] were used in the fitting process.
The constant term in equation (24) plays the role of the cosmological constant and we investigate the extra term from the phenomenological point of view.

3. Phenomenology

For standard cosmology in a "matter dominated" era,

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3, \quad (26)$$

where Ω_Λ is the fraction of vacuum energy, and the matter fraction, Ω_M, is interpreted as $\Omega_M = \Omega_{VM} + \Omega_{DM}$ where Ω_{VM} is the fraction of visible matter and Ω_{DM} is the fraction of dark matter. For the complex JBD model, the Friedmann equation becomes

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Delta \left(\frac{a_0}{a}\right)^6. \quad (27)$$

We make the variable transformation and define

$$u = \sqrt{\Omega_\Lambda}\left(\frac{a}{a_0}\right)^3 + \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}, \quad (28)$$

$$c = \sqrt{|\,\Omega_\Delta - \frac{\Omega_M^2}{4\Omega_\Lambda}\,|}, \quad (29)$$

$$\varkappa = sgn(\Omega_\Delta - \frac{\Omega_M^2}{4\Omega_\Lambda}). \quad (30)$$

so that Eq. (27) is put in differential form

$$3\sqrt{\Omega_\Lambda}H_0 dt = \frac{du}{\sqrt{(u^2 + \varkappa c^2)}}. \quad (31)$$

To test the viability of Equation (27) predicted by The complex JBD model, we have to compare the standard fit to Union compiled data [33] with a fit using equation (27) and H_0=71 km/sec/Mpc. With the constraint $\Omega_\Lambda + \Omega_M = 1$, the standard model fit has one free parameter which can be chosen as Ω_Λ.
The latest Type Ia supernova (SNIa) data sets in Table C2 [33] are taken by the Supernova Cosmology Project Group [33]. 414 SNIa supernova magnitude-redshift observations are compiled with "Union" and after selection cuts, it reduces to 307 Sne [33]. Luminosity distance, in Friedmann-Robertson-Walker Cosmology, is defined in [34]

$$D_L = c(1+z)\int \frac{dz'}{H(z')} \quad (32)$$

When we substitute the solution obtained in Eq.(24) into equation (32), the model estimated distance modulus is given as

$$D_L = \frac{c(1+z)}{H_o}\int \frac{dz'}{[\Omega_\Lambda + \Omega_M(1+z')^3 + \Omega_\Delta(1+z')^6]}. \quad (33)$$

Distance modulus can be written in terms of luminosity distance,

$$\mu = m - M = 5\log\left(\frac{D_L}{Mpc}\right) + 25, \quad (34)$$

where m is the apparent magnitude and M is the absolute magnitude. Equation (33) was put into Equation (34), and three different cases for $\varkappa$ were analyzed. Three fits of interest are:

$$1^\circ)\varkappa = -1, \Omega_\Delta = 0$$

standard cosmology fit with $\Omega_\Lambda + \Omega_M = 1$ gives,

$$\left(\frac{H}{H_0}\right)^2 = 0.745 + 0.255\left(\frac{a_0}{a}\right)^3, \quad (35)$$

with $\chi^2/d.o.f. = 1.45$. The scale factor and lifetime of universe are related by

$$\left(\frac{a}{a_0}\right)^3 = \frac{\Omega_M}{2\Omega_\Lambda}(\cosh(3H_0\sqrt{\Omega_\Lambda}t) - 1) \quad (36)$$

$$2^\circ)\varkappa = +1$$

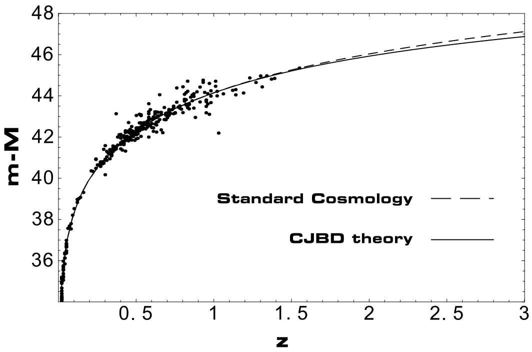

Figure 1. Magnitude vs. redshift graph

fit to the complex JBD model with $\Omega_M = 0$ and $\Omega_\Lambda + \Omega_\Delta = 1$ gives,

$$\left(\frac{H}{H_0}\right)^2 = 0.938 + 0.062\left(\frac{a_0}{a}\right)^6 \tag{37}$$

with $\chi^2/d.o.f. = 1.43$. The scale factor and lifetime of the universe are related by

$$\left(\frac{a}{a_0}\right)^3 = \sqrt{\frac{\Omega_\Delta}{\Omega_\Lambda}}\sinh(3H_0\sqrt{\Omega_\Lambda}t). \tag{38}$$

$3^\circ)\varkappa = 0$,

fit to the complex JBD model with matter, $\Omega_\Lambda + \Omega_M + \Omega_\Delta = 1$ and $\Omega_\Delta = \frac{\Omega_M^2}{4\Omega_\Lambda}$ so that equation (27) becomes

$$\frac{H}{H_0} = \sqrt{\Omega_\Lambda} + \sqrt{\Omega_\Delta}(\frac{a_0}{a})^3 \tag{39}$$

which gives

$$\left(\frac{H}{H_0}\right)^2 = 0.790 + 0.200\left(\frac{a_0}{a}\right)^3 + 0.010\left(\frac{a_0}{a}\right)^6, \tag{40}$$

with $\chi^2/d.o.f. = 1.43$. This condition makes the r.h.s. of the Friedmann equation a perfect square and gives the best fit to supernovae Union compiled data sets. The scale factor and lifetime of the universe are now related by,

$$\left(\frac{a}{a_0}\right)^3 = \frac{\Omega_M}{2\Omega_\Lambda}(\exp(3H_0\sqrt{\Omega_\Lambda}t) - 1). \tag{41}$$

The fit for equation (37) is interesting, but unacceptable since the model estimated lifetime is approximately 10 Gyrs, a value which is small compared to the observations [32]. In Fig. 1, the standard cosmology fit using Eq. (35) and the complex JBD fit Eq. (40) to Union compiled data sets [33] are shown. We conclude that the extra term from the change of the JBD field's phase, which must be small, indicates a slightly better fit. The difference between the two models will be seen for high redshift observations. Similarly to the addition of the cosmological constant in ΛCDM, the cosmological reason for obtaining the phase term in The JBD theory can be investigated.

4. Conclusion

A complex JBD model which, in addition to the constant term, makes a contribution of a $\frac{1}{a^6}$ term to the Friedmann Equation, fits the Supernovae data accurately. It is clear from (25), which is valid for $\rho_M = p_M = 0$, that this term is a natural component of dark energy. We conclude that the complex JBD model explains the evolution of the universe with a slightly better fit than the standard cosmology. It should be noted, however, that the lifetime of the Universe from the complex JBD model was found to be approximately 12 Gyrs, which is smaller than the experimental age of the Universe, 13.8 Gyrs [32]. Moreover, the complex scalar field is largely responsible for the expansion behavior of the Universe, and its phase behaves as a phenomenological extreme density term. As the density due to this complex phase increases, a smaller model-based age is determined. In standard cosmology, the $\frac{1}{a^6}$ term would be obtained if the kinetic term of a scalar field dominated the energy-momentum tensor. In our model, this is also how it mathematically arises. The physical interpretation, however, is that it is a component of dark energy, as a typical feature of CJBD theory, and its presence may be determined by more accurate measurements of the Supernova data near $z \approx 3$.

As we remarked in the Introduction, a complex scalar field is equivalent to two real fields with a same mass in our model, and thus considering two real fields instead of one contributes a $\frac{1}{a^6}$ term to the Friedmann equation. One interesting question is whether there will be additional contributions if more than two real fields are included in the model. We have checked that this does not occur so that the contribution is $\frac{1}{a^6}$ when the number of scalar fields with the same mass is greater or equal to two.

We have mentioned that, if one considers the Schwarzschild solution with constant phase, χ, the PPN parameters are exactly the same as the real field case. This point is under further investigation for the case of non-constant χ.

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Appendix A: STABILITY ANALYSIS

To test the stability of our vacuum solution we write the variables with small perturbations as

$$a = a(1 + \epsilon\eta) \tag{A1}$$

$$\varphi = \varphi(1 + \epsilon\psi) \tag{A2}$$

where η and $\psi = \psi_R + i\psi_I$ are perturbations and ϵ is small. Inserting these variables into Eq. (13)-(16), and after neglecting the higher order terms in ϵ, we obtain four homogeneous differential equations for the three functions η, ψ_R, and ψ_I and their derivatives in matrix form,

$$\begin{pmatrix} (6H+6F_1) & (6H-4\omega F_1) & -4\omega F_2 & 0 & 0 & 0 \\ (6H+4F_1) & (4\omega F_1+8F_1+4H) & 4\omega F_2 & 2 & 0 & 2 \\ (6\omega F_1-12H) & (4\omega F_1+6\omega H) & -4\omega F_2 & 2\omega & 0 & -3 \\ 6\omega F_2 & 4\omega F_2 & (4\omega F_1+6\omega H) & 0 & 2\omega & 0 \end{pmatrix} \begin{pmatrix} \dot{\eta} \\ \dot{\psi}_R \\ \dot{\psi}_I \\ \ddot{\psi}_R \\ \ddot{\psi}_I \\ \ddot{\eta} \end{pmatrix} = 0. \tag{A3}$$

We investigate the stability of this linear homogeneous system of equations by substituting

$$\eta = \eta_0 e^{\beta t} \tag{A4}$$

$$\psi_R = \psi_{R0} e^{\beta t} \tag{A5}$$

$$\psi_I = \psi_{I0} e^{\beta t} \tag{A6}$$

as solutions. This leads us to obtain four linear homogeneous equations for the three unknowns η_0, ψ_{R0}, and ψ_{I0}.

$$\begin{pmatrix} (6H+6F_1) & (6H-4\omega F_1) & -4\omega F_2 \\ (6H+4F_1+2\beta) & (4\omega F_1+8F_1+4H+2\beta) & 4\omega F_2 \\ (6\omega F_1-12H) & (4\omega F_1+6\omega H+2\omega\beta) & -4\omega F_2 \\ 6\omega F_2 & 4\omega F_2 & (4\omega F_1+6\omega H+2\omega\beta) \end{pmatrix} \begin{pmatrix} \eta_0 \\ \psi_{R0} \\ \psi_{I0} \end{pmatrix} = 0. \tag{A7}$$

This homogeneous system has a 4x3 matrix of coefficients. The condition that a non-trivial solution exists is that the rank of the matrix of coefficients is at most two. All 3x3 subdeterminants (A_{ij}) must be zero in order to obtain a non-trivial solution for β.
We thus obtain four equations for the four 3x3 subdeterminants, and arrange them in powers of β. One of these equation is cubic polynomial in β, whereas three are quadratic and given by

$$A_{i1}\beta^2 + A_{i2}\beta + A_{i3} = 0 \tag{A8}$$

where $i = 1, 2, 3$. The components of the matrix of subdeterminants are calculated as

$$\begin{aligned} A_{11} &= (3+2\omega) \\ A_{12} &= (27H + 18F_1 + 18\omega H + 12\omega F_1) \\ A_{13} &= (36\omega H^2 + 54F_1^2 + 48\omega H F_1 + 16\omega F_1^2 \\ &\quad + 72HF_1 + 24F_1^2) \\ A_{21} &= (2\omega^2 F_1 + 3\omega F_1) \\ A_{22} &= (2\omega F_2^2 + 12\omega^2 F_1 H - 3H^2\omega + 12\omega^2 F_1^2 + 27\omega H F_1 \\ &\quad + 12F_1^2 + 6\omega F_1^2) \\ A_{23} &= (6\omega H F_2^2 + 4\omega F_2^2 F_1 + 36\omega^2 H^2 F_1 - 9H^3\omega \\ &\quad + 114\omega H F_1^2 + 30\omega H^2 F_1 + 20\omega^2 F_1^3 + 24F_1^3\omega) \\ A_{31} &= (-2\omega^2 F_1 + 5\omega H + 2\omega F_1) \\ A_{32} &= (4\omega^2 F_2^2 + 4\omega F_2^2 + 4\omega^2 F_2^2 + 12\omega^2 H^2 + 6\omega H F_1 \\ &\quad + 21\omega H^2) \\ A_{33} &= (8\omega^2 F_2^2 F_1 + 14\omega F_2^2 H + 8\omega F_2^2 F_1 + 36\omega^2 F_1^2 H \\ &\quad + 16\omega F_1^3 + 18\omega^2 H^3 - 18\omega^2 H^2 F_1 + 42\omega H^2 F_1 \\ &\quad - 20\omega F_1^2 H + 36\omega H^3). \end{aligned} \tag{A9}$$

For a solution for β to exist the determinant of the matrix A must be zero.
The determinant of this matrix of coefficients ($detA$) is nonzero, so we can conclude that there is no solution for β and it means that the solution is stable.

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