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Existence of strongly symmetrical weakly pandiagonal graeco-latin squares

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Abstract: A graeco-latin square is a pair of orthogonal latin squares. It is a design of experiment in which the experimental units are grouped in three different ways. In this paper, constructions of a pair of orthogonal latin squares which are both strongly symmetrical and weakly pandiagonal are investigated. As a result, it is proved that there exists a pair of strongly symmetrical weakly pandiagonal orthogonal latin square of order n if and only if $n > 4$ and $n \equiv 0, 1, 3 \pmod{4}$ with only one possible exception for $n = 12$.

Keywords: Latin square, graeco, orthogonal, strongly symmetrical, weakly pandiagonal

1 Introduction

A *Latin square* of order n , denoted by $LS(n)$, is an $n \times n$ array such that every row and every column is a permutation of an n -set S . A *transversal* in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A *diagonal* Latin square is a Latin square with the additional property that the main diagonal and back diagonal are both transversals. Two $LS(n)$ s are called *orthogonal*, denoted by $OLS(n)$, if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A pair of orthogonal latin squares is also called a *graeco-latin square* because Euler ([7]) used Greek letters for one square of the pair and latin letters for the other. See also [5] for overview of the Latin squares.

In 1776, Euler presented a paper in which he constructed magic squares of orders 3, 4, and 5 from orthogonal latin squares. He posed the question for order 6, now known as Eulers 36 Officers Problem. He conjectured that no solution exists for order 6. Indeed he conjectured further that there exist orthogonal latin squares of all orders n except when $n \equiv 2 \pmod{4}$ ([7, 9]). Much later, Bose, Shrikhande and Parker [1] proved that the Euler conjecture was false for all orders n of the form $4k + 2$ except $n = 2$ or 6. Much shorter disproofs of Euler's conjecture have been obtained. See, for example, [4, 15].

Wallis and Zhu[11], Heinrich and Hilton[10], Brown, Cherry, Most, Most, Parker and Wallis [2] investigated the existence of orthogonal diagonal latin squares. They proved that there exists a pair of diagonal $OLS(n)$ s if and only if $n \notin \{2, 3, 6\}$.

An $LS(n)$ is called *self-orthogonal* if it is orthogonal to its transpose. Let n be a positive integer. Denote $I_n = \{0, 1, \dots, n-1\}$. An $LS(n)$ over I_n $A = (a_{i,j}) (i, j \in I_n)$ is called *strongly symmetrical* if $a_{i,j} + a_{n-1-i, n-1-j} = n-1$, $i, j \in I_n$. Danhof, Phillips and Wallis [6] investigated the existence of strongly symmetrical $OLS(n)$ s. Du and Cao[8] investigated the existence of strongly symmetrical self-orthogonal $LS(n)$. Cao and Li [3] proved that there is a strongly symmetrical self-orthogonal $LS(n)$ if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$. It results in the following.

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Lemma 1.1. ([3]) *A pair of strongly symmetrical OLS(n) exists if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$.*

Given positive integers a, n , we denote $a \pmod{n}$ by $\langle a \rangle_n$ for convenience. For a matrix A of order n and $k \in I_n$, we call the elements $a_{i, \langle k+i \rangle_n}, i \in I_n$ the k -th right pandiagonal and $a_{i, \langle k-i \rangle_n}, i \in I_n$ the k -th left pandiagonal. Clearly, 0-th right pandiagonal is the main diagonal and the $(n-1)$ -th left pandiagonal is the back diagonal. Let A be an LS(n) over I_n . A is weakly pandiagonal if the sum of the n numbers in each pandiagonal is the same. For example, the following is a pair of weakly pandiagonal OLS(8).

$$D = \begin{pmatrix} 0 & 3 & 6 & 5 & 4 & 7 & 2 & 1 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 & 0 \\ 5 & 6 & 3 & 0 & 1 & 2 & 7 & 4 \\ 4 & 7 & 2 & 1 & 0 & 3 & 6 & 5 \\ 2 & 1 & 4 & 7 & 6 & 5 & 0 & 3 \\ 3 & 0 & 5 & 6 & 7 & 4 & 1 & 2 \\ 7 & 4 & 1 & 2 & 3 & 0 & 5 & 6 \\ 6 & 5 & 0 & 3 & 2 & 1 & 4 & 7 \end{pmatrix}, \quad D' = \begin{pmatrix} 0 & 1 & 5 & 4 & 2 & 3 & 7 & 6 \\ 3 & 2 & 6 & 7 & 1 & 0 & 4 & 5 \\ 6 & 7 & 3 & 2 & 4 & 5 & 1 & 0 \\ 5 & 4 & 0 & 1 & 7 & 6 & 2 & 3 \\ 4 & 5 & 1 & 0 & 6 & 7 & 3 & 2 \\ 7 & 6 & 2 & 3 & 5 & 4 & 0 & 1 \\ 2 & 3 & 7 & 6 & 0 & 1 & 5 & 4 \\ 1 & 0 & 4 & 5 & 3 & 2 & 6 & 7 \end{pmatrix}.$$

Xu and Lu [12] introduced weakly pandiagonal OLS(n) to construct pandiagonal magic squares. They proved that a weakly pandiagonal self-orthogonal LS(n) exists if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \not\equiv 3, 6 \pmod{9}$.

Zhang, Li and Lei [13] investigated a pair of weakly pandiagonal OLS(n). They proved the following.

Lemma 1.2. ([13]) *A pair of weakly pandiagonal OLS(n) exists if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$.*

We use SPOLS(n) to denote strongly symmetrical weakly pandiagonal OLS(n). Clearly, the above (D, D') is a pair of SPOLS(8). In this paper, the existence of a pair of SPOLS(n) is investigated. The construction of SPOLS(n) is provided in Section 2 and in section 3 the following is proved.

Theorem 1.3. *A pair of SPOLS(n) exists if and only if $n > 4$ and $n \equiv 0, 1, 3 \pmod{4}$ with only one possible exception for $n = 12$.*

2 Constructions for SPOLS(n)

Given an $m \times n$ matrix A and an $r \times s$ matrix B , the Kronecker Product $A \otimes B$ is an $mr \times ns$ matrix given as follows.

$$A \otimes B = \begin{pmatrix} a_{0,0}B & a_{0,1}B & \cdots & a_{0,n-1}B \\ a_{1,0}B & a_{1,1}B & \cdots & a_{1,n-1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,0}B & a_{m-1,1}B & \cdots & a_{m-1,n-1}B \end{pmatrix}.$$

Denote $J_{m \times n}$ an $m \times n$ matrix with all elements 1s. A $J_{n \times n}$ is denoted by J_n . The Kronecker Product is used to get the following construction.

Construction 2.1. *If there exists a pair of SPOLS(m) and there exists a pair of SPOLS(n), then there exists a pair of SPOLS(mn).*

Proof Suppose that (A, B) is a pair of SPOLS(m) over I_m and (C, D) is a pair of SPOLS(n) over I_n . Let

$$E = (e_{i,j}) = nA \otimes J_n + J_m \otimes C, \quad F = (f_{i,j}) = nB \otimes J_n + J_m \otimes D,$$

where

$$e_{i,j} = na_{u,s} + c_{v,t}, \quad f_{i,j} = nb_{u,s} + d_{v,t}, \\ i = nu + v, \quad j = ns + t, \quad u, s \in I_m, \quad v, t \in I_n.$$

In [13] Construction 2.1 proved that (E, F) is a pair of weakly pandiagonal OLS(mn). We shall prove that E, F are also strongly symmetrical.

Let $i' = mn - 1 - i, j' = mn - 1 - j, u' = m - 1 - u, v' = n - 1 - u, s' = m - 1 - s, t' = n - 1 - t$. Clearly, $i' = nu' + v', j' = ns' + t'$. So we have

$$\begin{aligned} e_{i,j} + e_{i',j'} &= na_{u,s} + c_{v,t} + na_{u',s'} + c_{v',t'} \\ &= n(a_{u,s} + a_{u',s'}) + (c_{v,t} + c_{v',t'}) \\ &= n(m - 1) + n - 1 = nm - 1. \end{aligned}$$

Thus E is strongly symmetrical. Similarly, F is also strongly symmetrical. So (E, F) is a pair of SPOLS(mn). $\square$

Let m be an even integer and n be an integer. Let $T = \{T_0, T_1, \dots, T_{m-1}\}$ be a partition of I_{mn} such that $|T_i| = n, i \in I_m$. Denote $S = \frac{mn(mn-1)}{2}$. Given a rational number x , we use $\lfloor x \rfloor$ to denote the largest integer a such that $a \leq x$ as usual.

Construction 2.2. Let (D, D') be a pair of strongly symmetrical OLS(m) over I_m , and (M, M') be a pair of strongly symmetrical OLS(n) over I_n , where $D = (d_{u,v}), D' = (d'_{u,v}), M = (m_{i,j}), M' = (m'_{i,j})$. Let $\varphi_{u,v}, \varphi'_{u,v}$ be a bijection from I_n to $T_v, u, v \in I_m$. Let

$$\begin{aligned} A &= (A_{u,v}), \quad A_{u,v} = (a_{u,v}(i, j))_{n \times n}, \quad a_{u,v}(i, j) = \varphi_{u,d_{u,v}}(m_{i,j}), \\ B &= (B_{u,v}), \quad B_{u,v} = (b_{u,v}(i, j))_{n \times n}, \quad b_{u,v}(i, j) = \varphi'_{u,d'_{u,v}}(m'_{i,j}), \end{aligned}$$

where $u, v \in I_m, i, j \in I_n$. If $\varphi_{u,v}, \varphi'_{u,v}$ satisfy

$$(L1) \quad \varphi_{u,v}(t) + \varphi_{m-1-u, m-1-v}(n-1-t) = mn-1,$$

$$(L2) \quad \varphi'_{u,v}(t) + \varphi'_{m-1-u, m-1-v}(n-1-t) = mn-1,$$

Then (A, B) is a pair of strongly symmetrical OLS(mn) over I_{mn} . Further, if $\varphi_{u,v}$ satisfies

$$(L3) \quad \sum_{u \in I_m} \sum_{i \in I_n} \varphi_{u,d_{u, \langle v+u+\lfloor \frac{j+i}{n} \rfloor \rangle_m}}(m_{i, \langle j+i \rangle_n}) = S,$$

$$(L4) \quad \sum_{u \in I_m} \sum_{i \in I_n} \varphi_{u,d_{u, \langle v-u-\lfloor \frac{j-i}{n} \rfloor \rangle_m}}(m_{i, \langle j-i \rangle_n}) = S,$$

and $\varphi'_{u,v}$ satisfies

$$(L5) \quad \sum_{u \in I_m} \sum_{i \in I_n} \varphi'_{u,d'_{u, \langle v+u+\lfloor \frac{j+i}{n} \rfloor \rangle_m}}(m'_{i, \langle j+i \rangle_n}) = S,$$

$$(L6) \quad \sum_{u \in I_m} \sum_{i \in I_n} \varphi'_{u,d'_{u, \langle v-u-\lfloor \frac{j-i}{n} \rfloor \rangle_m}}(m'_{i, \langle j-i \rangle_n}) = S,$$

then (A, B) is a pair of SPOLS(mn) over I_{mn} .

Proof First we prove that A, B are LS(mn). Let $u \in I_m, i \in I_n$. We consider the entries $a_{u,v}(i, j)$ and $a_{u,v'}(i, j')$ of the $(nu + i)$ -th row of A , where $v, v' \in I_m, j, j' \in I_n$, and $(v, j) \neq (v', j')$. Since D is a latin square, if $v \neq v'$ then $d_{u,v} \neq d_{u,v'}$, hence $T_{d_{u,v}} \cap T_{d_{u,v'}} = \emptyset$. So, $\varphi_{u,d_{u,v}}(m_{i,j}) \neq \varphi_{u,d_{u,v'}}(m_{i,j'})$. Thus $a_{u,v}(i, j) \neq a_{u,v'}(i, j')$. If $v = v', j \neq j'$, then the image of $\varphi_{u,d_{u,v}}$ and $\varphi_{u,d_{u,v'}}$ fall into the same set $T_{d_{u,v}}$. Since M is latin square, $m_{i,j} \neq m_{i,j'}$, thus $\varphi_{u,d_{u,v}}(m_{i,j}) \neq \varphi_{u,d_{u,v'}}(m_{i,j'})$. So, $a_{u,v}(i, j) \neq a_{u,v'}(i, j')$. Therefore each row of A is a permutation of I_{mn} . Similarly, each column of A is also a permutation of I_{mn} . Thus A is an LS(mn) over I_{mn} . Similarly, B is also an LS(mn) over I_{mn} .

Now we shall prove that A, B are orthogonal. Let

$$a_{u_1, v_1}(i_1, j_1) b_{u_1, v_1}(i_1, j_1) = a_{u_2, v_2}(i_2, j_2) a_{u_2, v_2}(i_2, j_2),$$

where $u_1, u_2, v_1, v_2 \in I_m, i_1, i_2, j_1, j_2 \in I_n$. Then

$$\begin{aligned}\varphi_{u_1, d_{u_1, v_1}}(m_{i_1, j_1}) &= \varphi_{u_2, d_{u_2, v_2}}(m_{i_2, j_2}), \\ \varphi'_{u_1, d'_{u_1, v_1}}(m'_{i_1, j_1}) &= \varphi'_{u_2, d'_{u_2, v_2}}(m'_{i_2, j_2}).\end{aligned}$$

Note that $\varphi_{u_i, d_{u_i, v_i}}$ and $\varphi'_{u_i, d'_{u_i, v_i}}$ are bijection from I_n to $T_{d_{u_i, v_i}}$ and $T_{d'_{u_i, v_i}}$, respectively, $i = 1, 2$. So, $T_{d_{u_1, v_1}} = T_{d_{u_2, v_2}}$, $T_{d'_{u_1, v_1}} = T_{d'_{u_2, v_2}}$. Thus $d_{u_1, v_1} = d_{u_2, v_2}$, $d'_{u_1, v_1} = d'_{u_2, v_2}$. Since D, D' are orthogonal, $u_1 = u_2$, $v_1 = v_2$. Since φ, φ' are bijection, we have

$$m_{i_1, j_1} = m_{i_2, j_2}, m'_{i_1, j_1} = m'_{i_2, j_2}.$$

Since M, M' are orthogonal, we have $i_1 = i_2$, $j_1 = j_2$. Thus A, B are orthogonal.

For any $u, v \in I_m$, $i, j \in I_n$, let $u' = m - 1 - u$, $v' = m - 1 - v$, $i' = n - 1 - i$, $j' = n - 1 - j$. Since D, M are strongly symmetrical, we get $d_{u, v} + d_{u', v'} = m - 1$, $m_{i, j} + m_{i', j'} = n - 1$. By the condition (L1) we have

$$\begin{aligned}a_{u, v}(m_{i, j}) + a_{u', v'}(m_{i', j'}) &= \varphi_{u, d_{u, v}}(m_{i, j}) + \varphi_{u', d_{u', v'}}(m_{i', j'}) \\ &= \varphi_{u, d_{u, v}}(m_{i, j}) + \varphi_{u', m-1-d_{u, v}}(n-1-m_{i, j}) \\ &= mn - 1.\end{aligned}$$

Thus A is strongly symmetrical. Similar, B is also strongly symmetrical by the condition (L2).

Now we consider the pandiagonals of A, B . For any $s \in I_{mn}$, the s -th right pandiagonal of A consists of the $(k, \langle k + s \rangle_{mn})$ -entry of A , $k \in I_{mn}$. Let $k = un + i$, $s = vn + j$, where $u, v \in I_m$, $i, j \in I_n$. Note that $i + j = n \lfloor \frac{i+j}{n} \rfloor + \langle i + j \rangle_n$, we have

$$\begin{aligned}\langle k + s \rangle_{mn} &= \langle n(u + v) + i + j \rangle_{mn} \\ &= \langle n \langle u + v \rangle_m + i + j \rangle_{mn} \\ &= \langle n \langle u + v \rangle_m + n \lfloor \frac{i+j}{n} \rfloor + \langle i + j \rangle_n \rangle_{mn} \\ &= n \langle u + v + \lfloor \frac{i+j}{n} \rfloor \rangle_m + \langle i + j \rangle_n.\end{aligned}$$

So, the $(k, \langle k + s \rangle_{mn})$ -entry of A is

$$a_{u, \langle v+u+\lfloor \frac{i+j}{n} \rfloor \rangle_m}(i, \langle j + i \rangle_n),$$

which is just

$$\varphi_{u, d_{u, \langle v+u+\lfloor \frac{i+j}{n} \rfloor \rangle_m}}(m_{i, \langle j+i \rangle_n}).$$

The sum of each right pandiagonal of A is the same number S by (L3). Similarly, the sum of each left pandiagonal of A is also S by (L4). So A is weakly pandiagonal. By (L5) and (L6) B is also weakly pandiagonal. Thus (A, B) is a pair of SPOLS(mn) over I_{mn} . $\square$

Lemma 2.3. *There exists a pair of SPOLS(24).*

Proof Let (D, D') be the pair of strongly symmetrical OLS(8) given in the introduction. Let

$$M = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad M' = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Clearly, (M, M') is a pair of strongly symmetrical OLS(3). Let

$$\begin{aligned}L &= \left(0 \ 2 \ 6 \mid 3 \ 4 \ 1 \mid 8 \ 7 \ 10 \mid 11 \ 9 \ 5 \mid 21 \ 17 \ 23 \mid 20 \ 19 \ 22 \mid 15 \ 16 \ 13 \mid 14 \ 18 \ 12 \right), \\ L' &= \left(0 \ 2 \ 6 \mid 1 \ 3 \ 4 \mid 10 \ 8 \ 7 \mid 11 \ 9 \ 5 \mid 21 \ 17 \ 23 \mid 22 \ 19 \ 20 \mid 13 \ 16 \ 15 \mid 14 \ 18 \ 12 \right).\end{aligned}$$

Denote $L = (L_0 \ L_1 \ \cdots \ L_7)$, $L_v = (l_v(0) \ l_v(1) \ l_v(2))$, $v \in I_8$. Similarly, we denote L' .

For any $t \in I_3$, $v \in I_8$, let

$$\varphi_{u, v}(t) = \begin{cases} l_v(t), & u = 0, 1, 2, 3; \\ 8n - 1 - l_{7-v}(n - 1 - t), & u = 4, 5, 6, 7. \end{cases}$$

$$\varphi'_{u,v}(t) = \begin{cases} l'_v(t), & u = 0, 1, 2, 3; \\ 8n - 1 - l'_{7-v}(n - 1 - t), & u = 4, 5, 6, 7. \end{cases}$$

It is readily verified that φ, φ' satisfy (L1)-(L6) in Construction 2.2. Let

$$A = (A_{u,v}), \quad A_{u,v} = (a_{u,v}(i, j))_{n \times n}, \quad a_{u,v}(i, j) = \varphi_{u,d_{u,v}}(m_{i,j}), \\ B = (B_{u,v}), \quad B_{u,v} = (b_{u,v}(i, j))_{n \times n}, \quad b_{u,v}(i, j) = \varphi'_{u,d'_{u,v}}(m'_{i,j}),$$

where $u, v \in I_8, i, j \in I_3$. Thus we get a pair of SPOLS(24) as Construction 2.2, (A, B) as follows.

$$A = \begin{pmatrix} \begin{array}{ccc|ccc|ccc|ccc|ccc} 0 & 6 & 2 & 11 & 5 & 9 & 15 & 13 & 16 & 20 & 22 & 19 & 21 & 23 & 17 & 14 & 12 & 18 & 8 & 10 & 7 & 3 & 1 & 4 \\ 6 & 2 & 0 & 5 & 9 & 11 & 13 & 16 & 15 & 22 & 19 & 20 & 23 & 17 & 21 & 12 & 18 & 14 & 10 & 7 & 8 & 1 & 4 & 3 \\ 2 & 0 & 6 & 9 & 11 & 5 & 16 & 15 & 13 & 19 & 20 & 22 & 17 & 21 & 23 & 18 & 14 & 12 & 7 & 8 & 10 & 4 & 3 & 1 \\ \hline 3 & 1 & 4 & 8 & 10 & 7 & 14 & 12 & 18 & 21 & 23 & 17 & 20 & 22 & 19 & 15 & 13 & 16 & 11 & 5 & 9 & 0 & 6 & 2 \\ 1 & 4 & 3 & 10 & 7 & 8 & 12 & 18 & 14 & 23 & 17 & 21 & 22 & 19 & 20 & 13 & 16 & 15 & 5 & 9 & 11 & 6 & 2 & 0 \\ 4 & 3 & 1 & 7 & 8 & 10 & 18 & 14 & 12 & 17 & 21 & 23 & 19 & 20 & 22 & 16 & 15 & 13 & 9 & 11 & 5 & 2 & 0 & 6 \\ \hline 20 & 22 & 19 & 15 & 13 & 16 & 11 & 5 & 9 & 0 & 6 & 2 & 3 & 1 & 4 & 8 & 10 & 7 & 14 & 12 & 18 & 21 & 23 & 17 \\ 22 & 19 & 20 & 13 & 16 & 15 & 5 & 9 & 11 & 6 & 2 & 0 & 1 & 4 & 3 & 10 & 7 & 8 & 12 & 18 & 14 & 23 & 17 & 21 \\ 19 & 20 & 22 & 16 & 15 & 13 & 9 & 11 & 5 & 2 & 0 & 6 & 4 & 3 & 1 & 7 & 8 & 10 & 18 & 14 & 12 & 17 & 21 & 23 \\ \hline 21 & 23 & 17 & 14 & 12 & 18 & 8 & 10 & 7 & 3 & 1 & 4 & 0 & 6 & 2 & 11 & 5 & 9 & 15 & 13 & 16 & 20 & 22 & 19 \\ 23 & 17 & 21 & 12 & 18 & 14 & 10 & 7 & 8 & 1 & 4 & 3 & 6 & 2 & 0 & 5 & 9 & 11 & 13 & 16 & 15 & 22 & 19 & 20 \\ 17 & 21 & 23 & 18 & 14 & 12 & 7 & 8 & 10 & 4 & 3 & 1 & 2 & 0 & 6 & 9 & 11 & 5 & 16 & 15 & 13 & 19 & 20 & 22 \\ \hline 1 & 3 & 4 & 10 & 8 & 7 & 18 & 12 & 14 & 17 & 23 & 21 & 22 & 20 & 19 & 13 & 15 & 16 & 11 & 9 & 5 & 0 & 2 & 6 \\ 3 & 4 & 1 & 8 & 7 & 10 & 12 & 14 & 18 & 23 & 21 & 17 & 20 & 19 & 22 & 15 & 16 & 13 & 9 & 5 & 11 & 2 & 6 & 0 \\ 4 & 1 & 3 & 7 & 10 & 8 & 14 & 18 & 12 & 21 & 17 & 23 & 19 & 22 & 20 & 16 & 13 & 15 & 5 & 11 & 9 & 6 & 0 & 2 \\ \hline 0 & 2 & 6 & 11 & 9 & 5 & 13 & 15 & 16 & 22 & 20 & 19 & 17 & 23 & 21 & 18 & 12 & 14 & 10 & 8 & 7 & 1 & 3 & 4 \\ 2 & 6 & 0 & 9 & 5 & 11 & 15 & 16 & 13 & 20 & 19 & 22 & 23 & 21 & 17 & 12 & 14 & 18 & 8 & 7 & 10 & 3 & 4 & 1 \\ 6 & 0 & 2 & 5 & 11 & 9 & 16 & 13 & 15 & 19 & 22 & 20 & 21 & 17 & 23 & 14 & 18 & 12 & 7 & 10 & 8 & 4 & 1 & 3 \\ \hline 17 & 23 & 21 & 18 & 12 & 14 & 10 & 8 & 7 & 1 & 3 & 4 & 0 & 2 & 6 & 11 & 9 & 5 & 13 & 15 & 16 & 22 & 20 & 19 \\ 23 & 21 & 17 & 12 & 14 & 18 & 8 & 7 & 10 & 3 & 4 & 1 & 2 & 6 & 0 & 9 & 5 & 11 & 15 & 16 & 13 & 20 & 19 & 22 \\ 21 & 17 & 23 & 14 & 18 & 12 & 7 & 10 & 8 & 4 & 1 & 3 & 6 & 0 & 2 & 5 & 11 & 9 & 16 & 13 & 15 & 19 & 22 & 20 \\ \hline 22 & 20 & 19 & 13 & 15 & 16 & 11 & 9 & 5 & 0 & 2 & 6 & 1 & 3 & 4 & 10 & 8 & 7 & 18 & 12 & 14 & 17 & 23 & 21 \\ 20 & 19 & 22 & 15 & 16 & 13 & 9 & 5 & 11 & 2 & 6 & 0 & 3 & 4 & 1 & 8 & 7 & 10 & 12 & 14 & 18 & 23 & 21 & 17 \\ 19 & 22 & 20 & 16 & 13 & 15 & 5 & 11 & 9 & 6 & 0 & 2 & 4 & 1 & 3 & 7 & 10 & 8 & 14 & 18 & 12 & 21 & 17 & 23 \end{array} \end{pmatrix},$$

$$B = \begin{pmatrix} \begin{array}{ccc|ccc|ccc|ccc|ccc} 2 & 6 & 0 & 3 & 4 & 1 & 19 & 20 & 22 & 17 & 23 & 21 & 8 & 7 & 10 & 9 & 5 & 11 & 18 & 12 & 14 & 16 & 15 & 13 \\ 0 & 2 & 6 & 1 & 3 & 4 & 22 & 19 & 20 & 21 & 17 & 23 & 10 & 8 & 7 & 11 & 9 & 5 & 14 & 18 & 12 & 13 & 16 & 15 \\ 6 & 0 & 2 & 4 & 1 & 3 & 20 & 22 & 19 & 23 & 21 & 17 & 7 & 10 & 8 & 5 & 11 & 9 & 12 & 14 & 18 & 15 & 13 & 16 \\ \hline 9 & 5 & 11 & 8 & 7 & 10 & 16 & 15 & 13 & 18 & 12 & 14 & 3 & 4 & 1 & 2 & 6 & 0 & 17 & 23 & 21 & 19 & 20 & 22 \\ 11 & 9 & 5 & 10 & 8 & 7 & 13 & 16 & 15 & 14 & 18 & 12 & 1 & 3 & 4 & 0 & 2 & 6 & 21 & 17 & 23 & 22 & 19 & 20 \\ 5 & 11 & 9 & 7 & 10 & 8 & 15 & 13 & 16 & 12 & 14 & 18 & 4 & 1 & 3 & 6 & 0 & 2 & 23 & 21 & 17 & 20 & 22 & 19 \\ \hline 16 & 15 & 13 & 18 & 12 & 14 & 9 & 5 & 11 & 8 & 7 & 10 & 17 & 23 & 21 & 19 & 20 & 22 & 3 & 4 & 1 & 2 & 6 & 0 \\ 13 & 16 & 15 & 14 & 18 & 12 & 11 & 9 & 5 & 10 & 8 & 7 & 21 & 17 & 23 & 22 & 19 & 20 & 1 & 3 & 4 & 0 & 2 & 6 \\ 15 & 13 & 16 & 12 & 14 & 18 & 5 & 11 & 9 & 7 & 10 & 8 & 23 & 21 & 17 & 20 & 22 & 19 & 4 & 1 & 3 & 6 & 0 & 2 \\ \hline 19 & 20 & 22 & 17 & 23 & 21 & 2 & 6 & 0 & 3 & 4 & 1 & 18 & 12 & 14 & 16 & 15 & 13 & 8 & 7 & 10 & 9 & 5 & 11 \\ 22 & 19 & 20 & 21 & 17 & 23 & 0 & 2 & 6 & 1 & 3 & 4 & 14 & 18 & 12 & 13 & 16 & 15 & 10 & 8 & 7 & 11 & 9 & 5 \\ 20 & 22 & 19 & 23 & 21 & 17 & 6 & 0 & 2 & 4 & 1 & 3 & 12 & 14 & 18 & 15 & 13 & 16 & 7 & 10 & 8 & 5 & 11 & 9 \\ \hline 14 & 12 & 18 & 15 & 13 & 16 & 7 & 10 & 8 & 5 & 9 & 11 & 20 & 22 & 19 & 21 & 23 & 17 & 6 & 2 & 0 & 4 & 1 & 3 \\ 18 & 14 & 12 & 16 & 15 & 13 & 8 & 7 & 10 & 11 & 5 & 9 & 19 & 20 & 22 & 17 & 21 & 23 & 0 & 6 & 2 & 3 & 4 & 1 \\ 12 & 18 & 14 & 13 & 16 & 15 & 10 & 8 & 7 & 9 & 11 & 5 & 22 & 19 & 20 & 23 & 17 & 21 & 2 & 0 & 6 & 1 & 3 & 4 \\ \hline 21 & 23 & 17 & 20 & 22 & 19 & 4 & 1 & 3 & 6 & 2 & 0 & 15 & 13 & 16 & 14 & 12 & 18 & 5 & 9 & 11 & 7 & 10 & 8 \\ 17 & 21 & 23 & 19 & 20 & 22 & 3 & 4 & 1 & 0 & 6 & 2 & 16 & 15 & 13 & 18 & 14 & 12 & 11 & 5 & 9 & 8 & 7 & 10 \\ 23 & 17 & 21 & 22 & 19 & 20 & 1 & 3 & 4 & 2 & 0 & 6 & 13 & 16 & 15 & 12 & 18 & 14 & 9 & 11 & 5 & 10 & 8 & 7 \\ \hline 4 & 1 & 3 & 6 & 2 & 0 & 21 & 23 & 17 & 20 & 22 & 19 & 5 & 9 & 11 & 7 & 10 & 8 & 15 & 13 & 16 & 14 & 12 & 18 \\ 3 & 4 & 1 & 0 & 6 & 2 & 17 & 21 & 23 & 19 & 20 & 22 & 11 & 5 & 9 & 8 & 7 & 10 & 16 & 15 & 13 & 18 & 14 & 12 \\ 1 & 3 & 4 & 2 & 0 & 6 & 23 & 17 & 21 & 22 & 19 & 20 & 9 & 11 & 5 & 10 & 8 & 7 & 13 & 16 & 15 & 12 & 18 & 14 \\ \hline 7 & 10 & 8 & 5 & 9 & 11 & 14 & 12 & 18 & 15 & 13 & 16 & 6 & 2 & 0 & 4 & 1 & 3 & 20 & 22 & 19 & 21 & 23 & 17 \\ 8 & 7 & 10 & 11 & 5 & 9 & 18 & 14 & 12 & 16 & 15 & 13 & 0 & 6 & 2 & 3 & 4 & 1 & 19 & 20 & 22 & 17 & 21 & 23 \\ 10 & 8 & 7 & 9 & 11 & 5 & 12 & 18 & 14 & 13 & 16 & 15 & 2 & 0 & 6 & 1 & 3 & 4 & 22 & 19 & 20 & 23 & 17 & 21 \end{array} \end{pmatrix}.$$

□

We introduce the following definition.

Definition 1. Let L be a permutation of $0, 1, \dots, 4n - 1$. Denote $L = (L_0|L_1|L_2|L_3)$, where

$$L_i = (l_i(0) \ l_i(1) \ \dots \ l_i(n - 1)), \ i = 0, 1, 2, 3.$$

L is denoted by $S(4n)$ if it satisfies the following 3 conditions.

$$\begin{aligned} (R1) \quad & \{l_v(j) | j \in I_n\} = \{4n - 1 - l_{3-v}(j) | j \in I_n\}, & v = 0, 1, \\ (R2) \quad & l_0(j) + l_2(j) = l_1(j) + l_3(j), & j \in I_n, \\ (R3) \quad & l_0(j) + l_2(j) = l_0(n - 1 - j) + l_2(n - 1 - j), & j \in I_n, \end{aligned}$$

Construction 2.4. *If there is an $S(4n)$ and there is a pair of strongly symmetrical OLS(n), then there is a pair of SPOLS($4n$).*

Proof We use Construction 2.2 by taking $m = 4$. Let

$$D = \begin{pmatrix} 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}, \quad D' = \begin{pmatrix} 2 & 0 & 1 & 3 \\ 1 & 3 & 2 & 0 \\ 3 & 1 & 0 & 2 \\ 0 & 2 & 3 & 1 \end{pmatrix}.$$

Clearly, (D, D') a pair of strongly symmetrical OLS(4). Let (M, M') be a pair of strongly symmetrical OLS(n).

Let $L = (L_0 | L_1 | L_2 | L_3)$ be an $S(4n)$. Denote

$$L_v = (l_v(0) \ l_v(1) \ \cdots \ l_v(n-1)), \ v \in I_4,$$

For any $t \in I_n$, let

$$\varphi_{u,v}(t) = \begin{cases} l_v(t), & u = 0, 2; \\ 4n - 1 - l_{3-v}(n - 1 - t), & u = 1, 3. \end{cases} \quad (1)$$

and

$$\varphi'_{u,v}(t) = \varphi_{u,v}(t), \quad u, v \in I_4. \quad (2)$$

Let

$$T_v = \{l_v(j) | j \in I_n\}, \ v \in I_4, \quad T = \{T_0, T_1, T_2, T_3\}.$$

Since L is a permutation, T is a partition of I_{4n} . By (R1), $T_v = \{4n - 1 - x | x \in T_{3-v}\}$, $v \in I_4$.

For fixed $v \in I_4$, if t runs over I_n then $\varphi_{0,v}(t)$, $\varphi_{2,v}(t)$ run over T_v , and $\varphi_{1,v}(t)$, $\varphi_{3,v}(t)$ runs over $\{4n - 1 - x | x \in T_{3-v}\}$, which is exactly T_v . So, for any $u, v \in I_4$, $\varphi_{u,v}$ is a bijection from I_n to T_v . By (2) $\varphi'_{u,v}$ is also a bijection from I_n to T_v .

Let A, B be the matrices given by the Construction 2.2. We shall prove that φ and φ' satisfy (L1)-(L6) in Construction 2.2. So, (A, B) is a pair of SPOLS($4n$) by Construction 2.2.

In fact, by (1), for any $t \in I_n$,

$$\varphi_{0,v}(t) + \varphi_{3,3-v}(n - 1 - t) = l_v(t) + 4n - 1 - l_v(t) = 4n - 1.$$

Similarly, one can prove that $\varphi_{1,v}(t) + \varphi_{2,3-v}(n - 1 - t) = 4n - 1$. Thus (L1) and (L2) in Construction 2.2 are satisfied.

For any $t \in I_n$, by (1) and (R2) and (R3) it is readily verified the following.

$$\varphi_{0,0}(t) + \varphi_{0,2}(t) = \varphi_{0,0}(n - 1 - t) + \varphi_{0,2}(n - 1 - t), \quad (3)$$

$$\varphi_{0,1}(t) + \varphi_{0,3}(t) = \varphi_{0,0}(n - 1 - t) + \varphi_{0,2}(n - 1 - t), \quad (4)$$

$$\varphi_{0,1}(t) + \varphi_{0,3}(t) = \varphi_{0,1}(n - 1 - t) + \varphi_{0,3}(n - 1 - t). \quad (5)$$

By (1) we have

$$\varphi_{0,v}(t) = 4n - 1 - \varphi_{1,3-v}(n - 1 - t) = \varphi_{2,v}(t) = 4n - 1 - \varphi_{3,3-v}(n - 1 - t).$$

By (3) we have

$$\varphi_{0,0}(t) + \varphi_{1,1}(t) + \varphi_{2,2}(t) + \varphi_{3,3}(t) = 2(4n - 1), t \in I_n.$$

That is $\sum_{u \in I_4} \varphi_{u,d_{u,u}}(t) = 2(4n - 1), t \in I_n$. Similarly, by (4) and (5) we have

$$\sum_{u \in I_4} \varphi_{u,d_{u,\langle v+u \rangle_4}}(t) = 2(4n - 1), v = 1, 2, 3, t \in I_n.$$

Thus for any $v \in I_4, t \in I_n$, φ satisfies

$$\sum_{u \in I_4} \varphi_{u,d_{u,\langle v+u \rangle_4}}(t) = 2(4n - 1).$$

So,

$$\begin{aligned} \sum_{u \in I_4} \sum_{i \in I_n} \varphi_{u,d_{u,\langle v+u+\lfloor \frac{i+i}{n} \rfloor_m}}(m_{i,\langle j+i \rangle_n}) &= \sum_{\substack{i \in I_n \\ \lfloor \frac{j+i}{n} \rfloor=0}} (\sum_{u \in I_4} \varphi_{u,d_{u,\langle v+u \rangle_4}}(m_{i,\langle j+i \rangle_n})) + \sum_{\substack{i \in I_n \\ \lfloor \frac{j+i}{n} \rfloor=1}} (\sum_{u \in I_4} \varphi_{u,d_{u,\langle \langle v+1 \rangle_4+u \rangle_4}}(m_{i,\langle j+i \rangle_n})) \\ &= \sum_{\substack{i \in I_n \\ \lfloor \frac{j+i}{n} \rfloor=0}} 2(4n - 1) + \sum_{\substack{i \in I_n \\ \lfloor \frac{j+i}{n} \rfloor=1}} 2(4n - 1) \\ &= 2n(4n - 1). \end{aligned}$$

It shows that the condition (L3) in Construction 2.2 is satisfied.

The condition (L4) in Construction 2.2 is satisfied by (3), (4) and (5). So (L5) and (L6) are satisfied by (2). $\square$

The following example is provided to explain Construction 2.4 and its proof.

Example 1. Let (D, D') be the same as in Construction 2.4. Let $M = D, M' = D'$. Then D, D', M, M' are strongly symmetrical. Let

$$L = (0 \ 3 \ 5 \ 6 | 2 \ 1 \ 7 \ 4 | 14 \ 13 \ 11 \ 8 | 12 \ 15 \ 9 \ 10).$$

Clearly L is an S(16). By Construction 2.4 we have

$$A = \begin{pmatrix} 0 & 6 & 3 & 5 & 12 & 10 & 15 & 9 & 2 & 4 & 1 & 7 & 14 & 8 & 13 & 11 \\ 5 & 3 & 6 & 0 & 9 & 15 & 10 & 12 & 7 & 1 & 4 & 2 & 11 & 13 & 8 & 14 \\ 6 & 0 & 5 & 3 & 10 & 12 & 9 & 15 & 4 & 2 & 7 & 1 & 8 & 14 & 11 & 13 \\ 3 & 5 & 0 & 6 & 15 & 9 & 12 & 10 & 1 & 7 & 2 & 4 & 13 & 11 & 14 & 8 \\ 11 & 13 & 8 & 14 & 7 & 1 & 4 & 2 & 9 & 15 & 10 & 12 & 5 & 3 & 6 & 0 \\ 14 & 8 & 13 & 11 & 2 & 4 & 1 & 7 & 12 & 10 & 15 & 9 & 0 & 6 & 3 & 5 \\ 13 & 11 & 14 & 8 & 1 & 7 & 2 & 4 & 15 & 9 & 12 & 10 & 3 & 5 & 0 & 6 \\ 8 & 14 & 11 & 13 & 4 & 2 & 7 & 1 & 10 & 12 & 9 & 15 & 6 & 0 & 5 & 3 \\ 12 & 10 & 15 & 9 & 0 & 6 & 3 & 5 & 14 & 8 & 13 & 11 & 2 & 4 & 1 & 7 \\ 9 & 15 & 10 & 12 & 5 & 3 & 6 & 0 & 11 & 13 & 8 & 14 & 7 & 1 & 4 & 2 \\ 10 & 12 & 9 & 15 & 6 & 0 & 5 & 3 & 8 & 14 & 11 & 13 & 4 & 2 & 7 & 1 \\ 15 & 9 & 12 & 10 & 3 & 5 & 0 & 6 & 13 & 11 & 14 & 8 & 1 & 7 & 2 & 4 \\ 7 & 1 & 4 & 2 & 11 & 13 & 8 & 14 & 5 & 3 & 6 & 0 & 9 & 15 & 10 & 12 \\ 2 & 4 & 1 & 7 & 14 & 8 & 13 & 11 & 0 & 6 & 3 & 5 & 12 & 10 & 15 & 9 \\ 1 & 7 & 2 & 4 & 13 & 11 & 14 & 8 & 3 & 5 & 0 & 6 & 15 & 9 & 12 & 10 \\ 4 & 2 & 7 & 1 & 8 & 14 & 11 & 13 & 6 & 0 & 5 & 3 & 10 & 12 & 9 & 15 \end{pmatrix}, B = \begin{pmatrix} 11 & 14 & 13 & 8 & 5 & 0 & 3 & 6 & 7 & 2 & 1 & 4 & 9 & 12 & 15 & 10 \\ 13 & 8 & 11 & 14 & 3 & 6 & 5 & 0 & 1 & 4 & 7 & 2 & 15 & 10 & 9 & 12 \\ 8 & 13 & 14 & 11 & 6 & 3 & 0 & 5 & 4 & 1 & 2 & 7 & 10 & 15 & 12 & 9 \\ 14 & 11 & 8 & 13 & 0 & 5 & 6 & 3 & 2 & 7 & 4 & 1 & 12 & 9 & 10 & 15 \\ 2 & 7 & 4 & 1 & 12 & 9 & 10 & 15 & 14 & 11 & 8 & 13 & 0 & 5 & 6 & 3 \\ 4 & 1 & 2 & 7 & 10 & 15 & 12 & 9 & 8 & 13 & 14 & 11 & 6 & 3 & 0 & 5 \\ 1 & 4 & 7 & 2 & 15 & 10 & 9 & 12 & 13 & 8 & 11 & 14 & 3 & 6 & 5 & 0 \\ 7 & 2 & 1 & 4 & 9 & 12 & 15 & 10 & 11 & 14 & 13 & 8 & 5 & 0 & 3 & 6 \\ 9 & 12 & 15 & 10 & 7 & 2 & 1 & 4 & 5 & 0 & 3 & 6 & 11 & 14 & 13 & 8 \\ 15 & 10 & 9 & 12 & 1 & 4 & 7 & 2 & 3 & 6 & 5 & 0 & 13 & 8 & 11 & 14 \\ 10 & 15 & 12 & 9 & 4 & 1 & 2 & 7 & 6 & 3 & 0 & 5 & 8 & 13 & 14 & 11 \\ 12 & 9 & 10 & 15 & 2 & 7 & 4 & 1 & 0 & 5 & 6 & 3 & 14 & 11 & 8 & 13 \\ 0 & 5 & 6 & 3 & 14 & 11 & 8 & 13 & 12 & 9 & 10 & 15 & 2 & 7 & 4 & 1 \\ 6 & 3 & 0 & 5 & 8 & 13 & 14 & 11 & 10 & 15 & 12 & 9 & 4 & 1 & 2 & 7 \\ 3 & 6 & 5 & 0 & 13 & 8 & 11 & 14 & 15 & 10 & 9 & 12 & 1 & 4 & 7 & 2 \\ 5 & 0 & 3 & 6 & 11 & 14 & 13 & 8 & 9 & 12 & 15 & 10 & 7 & 2 & 1 & 4 \end{pmatrix}.$$

It is readily verified that (A, B) is a pair of SPOLS(16).

Definition 2. Let H be an $m \times n$ matrix over I_{mn} consisting of distinct numbers. H is called a *row magic rectangle* if

$$\sum_{j=0}^{n-1} h_{i,j} = \frac{n(mn-1)}{2}, i \in I_m. \quad (6)$$

A row magic rectangle H is called *centre-complementary*, denoted by (m, n) -CCMR, if H satisfies

$$h_{i,j} + h_{m-1-i, n-1-j} = mn - 1, i \in I_m, j \in I_n. \quad (7)$$

See similar definition in [14].

Construction 2.5. Let $n \equiv 3 \pmod{6}$, $n \neq 3$, if there is a $(3, \frac{n}{3})$ -CCMR, then there is a pair of SPOLS(n).

Proof Let H be a $(3, \frac{n}{3})$ -CCMR over I_n . Define a permutation σ of I_n :

$$\sigma(i) = h_{\langle i \rangle_3, \lfloor \frac{i}{3} \rfloor}, i \in I_n. \quad (8)$$

Let U, V be matrices of order n over I_n , where

$$u_{i,j} = \sigma(\langle 2i + j + 1 \rangle_n), v_{i,j} = \sigma(\langle 2i - j \rangle_n), i, j \in I_n. \quad (9)$$

It is readily verified that (U, V) is a pair of OLS(n). We shall prove that U and V is a pair of SPOLS(n).

First we shall prove that U, V are weakly pandiagonal. Clearly, the left pandiagonals of U and the right pandiagonals of V are transversals. For the left pandiagonal of U , fixed $w \in I_n$, let $w = 3s + t$, $s \in I_{\frac{n}{3}}$, $t \in I_3$. By (6), (8) and (9) we have

$$\sum_{i=0}^{n-1} u_{i, \langle i+w \rangle_n} = \sum_{i=0}^{n-1} \sigma(\langle 2i + \langle i+w \rangle_n + 1 \rangle_n) = \sum_{i=0}^{n-1} \sigma(\langle 3(i+s) + t + 1 \rangle_n), \quad (10)$$

We have $\{\langle 3(i+s) + t + 1 \rangle_n | i \in I_n\} = 3\{3i + \langle t + 1 \rangle_3 | i \in I_{\frac{n}{3}}\}$. In fact, since $t + 1 = 3\lfloor \frac{t+1}{3} \rfloor + \langle t + 1 \rangle_3$ and there is an integer q such that $i + s + \lfloor \frac{t+1}{3} \rfloor = q \cdot \frac{n}{3} + \langle i + s + \lfloor \frac{t+1}{3} \rfloor \rangle_{\frac{n}{3}}$, we have

$$\begin{aligned} \{\langle 3(i+s) + t + 1 \rangle_n | i \in I_n\} &= \{\langle 3(i + s + \lfloor \frac{t+1}{3} \rfloor) + \langle t + 1 \rangle_3 \rangle_n | i \in I_n\} \\ &= \{\langle 3\langle i + s + \lfloor \frac{t+1}{3} \rfloor \rangle_{\frac{n}{3}} + \langle t + 1 \rangle_3 \rangle_n | i \in I_n\} \\ &= \{3\langle i + s + \lfloor \frac{t+1}{3} \rfloor \rangle_{\frac{n}{3}} + \langle t + 1 \rangle_3 | i \in I_n\} \\ &= 3\{3i + \langle t + 1 \rangle_3 | i \in I_{\frac{n}{3}}\} \end{aligned}$$

Thus the equality (10) is $3 \sum_{i=0}^{\frac{n}{3}-1} \sigma(3i + \langle t + 1 \rangle_3)$. It also equals to $3 \sum_{i=0}^{\frac{n}{3}-1} h_{\langle t+1 \rangle_3, i}$ by the equality (8). Further, it is just $\frac{n(n-1)}{2}$ by (6). So the sum of the n numbers of each right pandiagonal of U is the same. Similarly, the sum of the n numbers of each left pandiagonal of V is the same. Thus U, V are weakly pandiagonal.

Now we prove that U, V are strongly symmetrical. For any $i \in I_n$, let $q = \lfloor \frac{i}{3} \rfloor$, $r = \langle i \rangle_3$, then $i = 3q + r$, $q \in I_{\frac{n}{3}}$, $r \in I_3$. So we have $n - 1 - i = 3(\frac{n}{3} - 1 - q) + 2 - r$. Thus by (7) and (8) we have

$$\sigma(i) + \sigma(n - 1 - i) = h_{r,q} + h_{2-r, \frac{n}{3}-1-q} = n - 1.$$

for $i, j \in I_n$, we have

$$u_{n-1-i, n-1-j} = \sigma(\langle 2(n-1-i) + (n-1-j) + 1 \rangle_n) = \sigma(\langle n-1 - \langle 2i + j + 1 \rangle_n \rangle_n),$$

which is just $\sigma(n - 1 - \langle 2i + j + 1 \rangle_n)$. So,

$$u_{i,j} + u_{n-1-i, n-1-j} = \sigma(\langle 2i + j + 1 \rangle_n) + \sigma(n - 1 - \langle 2i + j + 1 \rangle_n) = n - 1.$$

Thus U is strongly symmetrical. Similarly, one can prove that V is strongly symmetrical. Therefore, (U, V) is a pair of SPOSL(n). $\square$

An example is provided to explain the construction 2.5.

Example 2. Let $n = 9$, and

$$H = \begin{pmatrix} 0 & 7 & 5 \\ 6 & 4 & 2 \\ 3 & 1 & 8 \end{pmatrix}.$$

Clearly, H is a $(3, 3)$ -CCMR. By Construction 2.5, define a permutation of σ over I_9 :

$$\sigma(i) = h_{\langle i \rangle_3, \lfloor \frac{i}{3} \rfloor}, \quad i \in I_9.$$

That is

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 6 & 3 & 7 & 4 & 1 & 5 & 2 & 8 \end{pmatrix}.$$

Let (U, V) be a pair of OLS(9) given by (9), where

$$u_{i,j} = \sigma(\langle 2i + j + 1 \rangle_9), \quad v_{i,j} = \sigma(\langle 2i - j \rangle_9), \quad i, j \in I_9.$$

So,

$$U = \begin{pmatrix} 6 & 3 & 7 & 4 & 1 & 5 & 2 & 8 & 0 \\ 7 & 4 & 1 & 5 & 2 & 8 & 0 & 6 & 3 \\ 1 & 5 & 2 & 8 & 0 & 6 & 3 & 7 & 4 \\ 2 & 8 & 0 & 6 & 3 & 7 & 4 & 1 & 5 \\ 0 & 6 & 3 & 7 & 4 & 1 & 5 & 2 & 8 \\ 3 & 7 & 4 & 1 & 5 & 2 & 8 & 0 & 6 \\ 4 & 1 & 5 & 2 & 8 & 0 & 6 & 3 & 7 \\ 5 & 2 & 8 & 0 & 6 & 3 & 7 & 4 & 1 \\ 8 & 0 & 6 & 3 & 7 & 4 & 1 & 5 & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 8 & 2 & 5 & 1 & 4 & 7 & 3 & 6 \\ 3 & 6 & 0 & 8 & 2 & 5 & 1 & 4 & 7 \\ 4 & 7 & 3 & 6 & 0 & 8 & 2 & 5 & 1 \\ 5 & 1 & 4 & 7 & 3 & 6 & 0 & 8 & 2 \\ 8 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 0 \\ 6 & 0 & 8 & 2 & 5 & 1 & 4 & 7 & 3 \\ 7 & 3 & 6 & 0 & 8 & 2 & 5 & 1 & 4 \\ 1 & 4 & 7 & 3 & 6 & 0 & 8 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 3 & 6 & 0 & 8 \end{pmatrix},$$

It is readily verified that (U, V) is a pair of SPOLS(9).

3 Proof of Theorem 1.3

In this section we consider the existence of a pair of SPOLS(n). There doesn't exist a pair of SPOLS(n) if $n \equiv 2 \pmod{4}$ by Lemma 1.1. To prove Theorem 1.3 it remains to check the following three cases: (1) $n \equiv 1, 5 \pmod{6}$; (2) $n \equiv 3 \pmod{6}$; (3) $n \equiv 0 \pmod{4}$.

Case 1. $n \equiv 1, 5 \pmod{6}$

Lemma 3.1. *There exists a pair of SPOLS(n) if $n \equiv 1, 5 \pmod{6}$.*

Proof Let $U = (u_{i,j})_{n \times n}$, $V = (v_{i,j})_{n \times n}$, where

$$u_{i,j} = \langle 2i + j + 1 \rangle_n, \quad v_{i,j} = \langle 2i - j \rangle_n, \quad i, j \in I_n.$$

(U, V) is a pair of OLS(n) over I_n . Now we prove that U, V are strongly symmetrical and weakly pandiagonal. It is enough to prove U is strongly symmetrical and weakly pandiagonal. Similarly we have V .

For any $i, j \in I_n$, we have

$$\begin{aligned} u_{i,j} + u_{n-1-i, n-1-j} &= \langle 2i + j + 1 \rangle_n + \langle 2(n-1-i) + (n-1-j) + 1 \rangle_n \\ &= \langle 2i + j + 1 \rangle_n + \langle 3n - 1 - (2i + j + 1) \rangle_n \\ &= \langle 2i + j + 1 \rangle_n + \langle n - 1 - \langle 2i + j + 1 \rangle_n \rangle_n \\ &= n - 1. \end{aligned}$$

So U is strongly symmetrical.

We consider the pandiagonals of U . For any $k \in I_n$, since $\gcd(3, n) = 1$, we have

$$\begin{aligned}\{u_{i, \langle i+k \rangle_n} | i \in I_n\} &= \{\langle 2i + \langle i+k \rangle_n + 1 \rangle_n | i \in I_n\} \\ &= \{\langle 3i + k + 1 \rangle_n | i \in I_n\} \\ &= \{\langle i + k + 1 \rangle_n | i \in I_n\} \\ &= I_n.\end{aligned}$$

It shows that each right pandiagonal of U is a transversal. Similarly, each left pandiagonal of U is also a transversal. Thus U is weakly pandiagonal. $\square$

Case 2. $n \equiv 3 \pmod{6}$

Lemma 3.2. *There is a $(3, n)$ -CCMR for odd $n \geq 3$.*

Proof Let n be odd, $n \geq 3$, denote $n = 4s + w$, $w \in \{3, 5\}$, $s \geq 0$. We perform induction on s . If $s = 0$, a $(3, 3)$ -CCMR is given by Example 2, and a $(3, 5)$ -CCMR is given below.

$$\begin{pmatrix} 0 & 13 & 6 & 12 & 4 \\ 3 & 9 & 7 & 5 & 11 \\ 10 & 2 & 8 & 1 & 14 \end{pmatrix}.$$

For $s \geq 0$, suppose that $A(s, w)$ is a $(3, 4s + w)$ -CCMR. Then the row sum of $A(s, w)$ is

$$(4s + w)(12s + 3w - 1)/2.$$

The sum of each pair of symmetrical elements is $12s + 3w - 1$. Let

$$L = \begin{pmatrix} 0 & 5 \\ 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad R(s, w) = \begin{pmatrix} 12s + 3w + 8 & 12s + 3w + 9 \\ 12s + 3w + 7 & 12s + 3w + 10 \\ 12s + 3w + 6 & 12s + 3w + 11 \end{pmatrix}.$$

Denote

$$A(s + 1, w) = (L \mid A(s, w) + 6J_{3 \times (4s+w)} \mid R(s, w)).$$

It is readily verified that the elements of $A(s + 1, w)$ run over I_{n+12} . By calculation, for any $i \in I_3$, the sum of the elements of the i -th row of $A(s + 1, w)$ is

$$\frac{n(3n-1)}{2} + 6n + 2(3n) + 22 = \frac{(4(s+1)+w)(12(s+1)+3w-1)}{2},$$

which is independent on i . The sum of each pair of symmetrical elements of $A(s + 1, w)$ is $12s + 3w + 11 = 12(s + 1) + 3w - 1$, which is also independent on i . Thus $A(s + 1, w)$ is a $(3, 4(s + 1) + w)$ -CCMR. There is a $(3, n)$ -CCMR for any odd $n \geq 3$ by induction. $\square$

Lemma 3.3. *There exists a pair of SPOLS(n) if $n \equiv 3 \pmod{6}$ and $n \neq 3$.*

Proof It follows by Construction 2.5 and Lemma 3.2. $\square$

Theorem 3.4. *There is a pair of SPOLS(n) for any odd $n \geq 5$.*

Proof It follows by Lemma 3.1 and Lemma 3.3. $\square$

Case 3. $n \equiv 0 \pmod{4}$

Let $n = 4k$, $k \geq 1$. We should deal with the case $k \equiv 2 \pmod{4}$ and the case $k \equiv 0, 1, 3 \pmod{4}$.

Lemma 3.5. *There is a pair of SPOLS($4k$) for any positive integer $k \equiv 2 \pmod{4}$.*

Proof The (D, D') given in Section 1 is a pair of SPOLS(8). Lemma 2.3 gives a pair of SPOLS(24). Thus there is a pair of SPOLS($4k$), $k = 2, 6$. For $k \equiv 2 \pmod{4}$, $k \geq 10$, there is a pair of SPOLS($\frac{k}{2}$) by Theorem 3.4. Since there is a pair of SPOLS(8), there is a pair of SPOLS($4k$) by Construction 2.1. $\square$

Now we consider the case $k \equiv 0, 1, 3 \pmod{4}$.

Lemma 3.6. *There is an $S(4k)$ if $k \equiv 0, 1, 3 \pmod{4}$ and $k \geq 4$.*

Proof There is an $S(4k)$ for $k = 4, 5, 7$ as follows.

$$\begin{aligned} k = 4 : L &= (0 \ 3 \ 5 \ 6 | 2 \ 1 \ 7 \ 4 | 14 \ 13 \ 11 \ 8 | 12 \ 15 \ 9 \ 10), \\ k = 5 : L &= (3 \ 7 \ 10 \ 11 \ 14 | 0 \ 17 \ 6 \ 18 \ 4 | 13 \ 19 \ 1 \ 15 \ 2 | 16 \ 9 \ 5 \ 8 \ 12), \\ k = 7 : L &= (1 \ 21 \ 4 \ 12 \ 17 \ 7 \ 3 | 9 \ 0 \ 8 \ 22 \ 11 \ 2 \ 13 | 18 \ 5 \ 27 \ 25 \ 14 \ 19 \ 16 | 10 \ 26 \ 23 \ 15 \ 20 \ 24 \ 6). \end{aligned}$$

Suppose that there is an $S(4k)$ for any $k \geq 4$, $k \equiv 0, 1, 3 \pmod{4}$,

$$L = (L_0 \ L_1 \ L_2 \ L_3),$$

where the j -position of L_i is $l_i(j)$, $i \in I_4, j \in I_k$. Let

$$H = (H_0 \ H_1 \ H_2 \ H_3)$$

be the $S(16)$ given above. We shall construct an $S(4(k+4))$. Let

$$\tilde{L} = (\tilde{L}_0 \ \tilde{L}_1 \ \tilde{L}_2 \ \tilde{L}_3),$$

where

$$\begin{aligned} \tilde{L}_0 &= (h_0(0), h_0(1), l_0(0) + 8, \dots, l_0(k-1) + 8, h_0(2), h_0(3)), \\ \tilde{L}_1 &= (h_1(0), h_1(1), l_1(0) + 8, \dots, l_1(k-1) + 8, h_1(2), h_1(3)), \\ \tilde{L}_2 &= (h_2(0) + 16, h_2(1) + 16, l_2(0) + 8, \dots, l_2(k-1) + 8, h_2(2) + 16, h_2(3) + 16), \\ \tilde{L}_3 &= (h_3(0) + 16, h_3(1) + 16, l_3(0) + 8, \dots, l_3(k-1) + 8, h_3(2) + 16, h_3(3) + 16). \end{aligned}$$

Clearly, the elements of $\tilde{L}$ run over $I_{4(k+4)}$, and it satisfies the conditions (R1)-(R3). Thus $\tilde{L}$ is an $S(4(k+4))$. The proof is completed by induction. $\square$

Example 3. By using the $S(16)$ given in Lemma 3.6 we can get an $S(32)$ as follows:

$$L = (0 \ 3 \ 8 \ 11 \ 13 \ 14 \ 5 \ 6 | 2 \ 1 \ 10 \ 9 \ 15 \ 12 \ 7 \ 4 | 30 \ 29 \ 22 \ 21 \ 19 \ 16 \ 27 \ 24 | 28 \ 31 \ 20 \ 23 \ 17 \ 18 \ 25 \ 26).$$

Lemma 3.7. *There is a pair of SPOLS($4k$) for $k \equiv 0, 1, 3 \pmod{4}$ and $k > 4$.*

Proof Let $k \equiv 0, 1, 3 \pmod{4}$, $k > 4$. There is a pair of strongly symmetrical OLS(k) by Lemma 1.1, and there is an $S(4k)$ by Lemma 3.6. Thus there is a pair of SPOLS($4k$) by Construction 2.4. $\square$

Theorem 3.8. *There is a pair of SPOLS($4k$) for $k \geq 2$ and $k \neq 3$.*

Proof The result follows from Lemma 3.5 and Lemma 3.7. $\square$

Lemma 3.9. *There is no SPOLS(4).*

Proof Suppose that there is a pair of SPOLS(4) over $I_4, (A, B)$. Let

$$U = 4A + B = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \\ a_3 & b_3 & a_4 & b_4 \\ c_3 & d_3 & c_4 & d_4 \end{pmatrix}.$$

Let

$$\begin{aligned} e &= a_1 + a_2 + a_3 + a_4, & f &= b_1 + b_2 + b_3 + b_4, \\ g &= c_1 + c_2 + c_3 + c_4, & h &= d_1 + d_2 + d_3 + d_4. \end{aligned}$$

Then $e + f = e + g = 60$, hence $f = g$. Since A and B are strongly symmetrical, we have $b_1 + c_4 = c_1 + b_4 = b_2 + c_3 = c_2 + b_3 = 15$. Thus $f + g = 60$. So, $f = g = 30$. On the other hand, since A is weakly pandiagonal, $b_2 + c_2 + b_3 + c_3 = b_2 + c_1 + b_3 + c_4 = 30$. So $c_2 + c_3 = c_1 + c_4$, $c_1 + c_2 + c_3 + c_4 = 2(c_1 + c_4) = 30$, it follows that $c_1 + c_4 = 15$. Previously we have proved that $c_1 + b_4 = 15$, so $c_4 = b_4$, a contradiction. Thus there is no SPOLS(4). $\square$

Proof of Theorem 1.3 The proof follows by Combining Theorem 3.4, Theorem 3.8 and Lemma 3.9.

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