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Some Hermite-Hadamard type inequalities for operator convex functions and positive maps

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Abstract: In this paper we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

Keywords: Jensen's inequality,Hermite-Hadamard inequality, Positive maps, Operator convex functions, Arithmetic mean-Geometric mean inequality

MSC: 47A63, 47A30, 15A60

1 Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A+\lambda B) \leq (\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see for instance [12] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function$f(t) = e^t$ is neither operator convex nor operator monotone.

In [4], see also [5, p. 60], we established the following Hermite-Hadamard type inequality for operator convex functions:

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Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$f\left(\frac{A+B}{2}\right) \le \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right] \tag{1.1}$$
$$\le \int_0^1 f((1-t)A+tB)\,dt$$
$$\le \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right] \le \frac{f(A)+f(B)}{2}.$$

For recent related results on operator Hermite-Hadamard type inequalities, see [1]-[2], [5]-[10] and [13].

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [3] (see also [12, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$$

for any $\lambda, \mu \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \le B \text{ implies } \Phi(A) \le \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \le A \le \beta 1_H$, then $\alpha 1_K \le \Phi(A) \le \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

The following Jensen's type result is well known [3]:

Theorem 2 (Davis-Choi-Jensen's Inequality). *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \le \Phi(f(A)). \tag{1.2}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.2) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \le \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1_H)f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \le \Psi(f(A)). \tag{1.3}$$

In this paper, motivated by the above results, we establish some inequalities of Hermite-Hadamard type for operator convex functions and positive maps. Applications for power function and logarithm are also provided.

2 Refinements of HH-Inequality

Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and two selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$. We know that Φ is continuous, see for instance [11, Proposition 2.8]. By taking the positive map Φ in (1.1) and using the continuity property of Φ, we have

$$\begin{aligned} \Phi\left(f\left(\frac{A+B}{2}\right)\right) &\leq \frac{1}{2}\left[\Phi\left(f\left(\frac{3A+B}{4}\right)\right) + \Phi\left(f\left(\frac{A+3B}{4}\right)\right)\right] \\ &\leq \int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt \\ &\leq \frac{1}{2}\left[\Phi\left(f\left(\frac{A+B}{2}\right)\right) + \frac{\Phi(f(A)) + \Phi(f(B))}{2}\right] \\ &\leq \frac{\Phi(f(A)) + \Phi(f(B))}{2}. \end{aligned} \tag{2.1}$$

If we write the inequality (2.1) for $\Phi(A)$ and $\Phi(B)$ then we also have

$$\begin{aligned} f\left(\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3\Phi(A)+\Phi(B)}{4}\right) + f\left(\frac{\Phi(A)+3\Phi(B)}{4}\right)\right] \\ &\leq \int_0^1 f\left((1-t)\Phi(A)+t\Phi(B)\right)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{\Phi(A)+\Phi(B)}{2}\right) + \frac{f(\Phi(A)) + f(\Phi(B))}{2}\right] \\ &\leq \frac{f(\Phi(A)) + f(\Phi(B))}{2}. \end{aligned} \tag{2.2}$$

It is then natural to ask how the following integrals

$$\int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt \text{ and } \int_0^1 f\left((1-t)\Phi(A)+t\Phi(B)\right)dt$$

do compare?

The following simple result holds:

Theorem 3. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\int_0^1 f\left((1-t)\Phi(A)+t\Phi(B)\right)dt \leq \int_0^1 \Phi\left(f\left((1-t)A+tB\right)\right)dt. \tag{2.3}$$

Proof. By (1.2) we have

$$f\left((1-t)\Phi(A)+t\Phi(B)\right) = f\left(\Phi\left(((1-t)A+tB)\right)\right) \leq \Phi\left(f\left((1-t)A+tB\right)\right)$$

for any $t \in [0, 1]$.

By integrating this inequality on $[0, 1]$ and using the continuity property of Φ we get the desired result (2.3). □

We define by $\mathfrak{P}_I[\mathcal{B}(H), \mathcal{B}(K)]$ the convex cone of all linear, positive maps Ψ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, namely $\Psi(1_H)$ is positive invertible operator in K.

Corollary 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and selfadjoint operators A and B with spectra in I. If $\Psi \in \mathfrak{P}_I [\mathcal{B}(H), \mathcal{B}(K)]$, then we have*

$$\Psi^{1/2}(1_H)\left(\int_0^1 f\left(\Psi^{-1/2}(1_H)((1-t)\Psi(A)+t\Psi(B))\Psi^{-1/2}(1_H)\right)dt\right)\Psi^{1/2}(1_H) \leq \int_0^1 \Psi(f((1-t)A+tB))\,dt. \tag{2.4}$$

Proof. If we write the inequality (2.3) for $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$, then we get

$$\int_0^1 f\left((1-t)\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)+t\Psi^{-1/2}(1_H)\Psi(B)\Psi^{-1/2}(1_H)\right)dt \leq \int_0^1 \Psi^{-1/2}(1_H)\Psi(f((1-t)A+tB))\Psi^{-1/2}(1_H)\,dt,$$

that can be written as

$$\int_0^1 f\left(\Psi^{-1/2}(1_H)((1-t)\Psi(A)+t\Psi(B))\Psi^{-1/2}(1_H)\right)dt \leq \Psi^{-1/2}(1_H)\left(\int_0^1 \Psi(f((1-t)A+tB))\,dt\right)\Psi^{-1/2}(1_H).$$

Finally, if we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$, then we get the desired result (2.4). □

The following representation result holds.

Lemma 1. *Let $f : I \to \mathbb{C}$ be a continuous function on the interval I and two selfadjoint operators A and B with spectra in I. Then for any $\lambda \in [0,1]$ we have the representation*

$$\int_0^1 f((1-t)A+tB)\,dt = (1-\lambda)\int_0^1 f[(1-t)((1-\lambda)A+\lambda B)+tB]\,dt + \lambda\int_0^1 f[(1-t)A+t((1-\lambda)A+\lambda B)]\,dt. \tag{2.5}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (2.5) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 f[(1-t)(\lambda B+(1-\lambda)A)+tB]\,dt = \int_0^1 f[((1-t)\lambda+t)B+(1-t)(1-\lambda)A]\,dt$$

and

$$\int_0^1 f[t(\lambda B+(1-\lambda)A)+(1-t)A]\,dt = \int_0^1 f[t\lambda B+(1-\lambda t)A]\,dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\,du$. Then

$$\int_0^1 f[((1-t)\lambda+t)B+(1-t)(1-\lambda)A]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 f[uB+(1-u)A]\,du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f[t\lambda B + (1-\lambda t)A]\,dt = \frac{1}{\lambda}\int_0^\lambda f[uB + (1-u)A]\,du.$$

Therefore

$$\begin{aligned}(1-\lambda)\int_0^1 f[(1-t)(\lambda B + (1-\lambda)A) + tB]\,dt + \lambda\int_0^1 f[t(\lambda B + (1-\lambda)A) + (1-t)A]\,dt \\ = \int_\lambda^1 f[uB + (1-u)A]\,du + \int_0^\lambda f[uB + (1-u)A]\,du \\ = \int_0^1 f[uB + (1-u)A]\,du\end{aligned}$$

and the identity (2.5) is proved. □

We have now the following generalization of (1.1):

Theorem 4. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0, 1]$ we have the inequalities*

$$\begin{aligned}f\left(\frac{A+B}{2}\right) &\le (1-\lambda)f\left[\frac{(1-\lambda)A + (1+\lambda)B}{2}\right] + \lambda f\left[\frac{(2-\lambda)A + \lambda B}{2}\right] \\ &\le \int_0^1 f((1-t)A + tB)\,dt \\ &\le \frac{1}{2}[f((1-\lambda)A + \lambda B) + (1-\lambda)f(B) + \lambda f(A)] \\ &\le \frac{f(A) + f(B)}{2}.\end{aligned} \tag{2.6}$$

Proof. Using the Hermite-Hadamard inequality (1.1) we have

$$\begin{aligned}f\left[\frac{(1-\lambda)A + (1+\lambda)B}{2}\right] &\le \int_0^1 f[(1-t)((1-\lambda)A + \lambda B) + tB]\,dt \\ &\le \frac{f((1-\lambda)A + \lambda B) + f(B)}{2}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}f\left[\frac{(2-\lambda)A + \lambda B}{2}\right] &\le \int_0^1 f[(1-t)A + t((1-\lambda)A + \lambda B)]\,dt \\ &\le \frac{f(A) + f((1-\lambda)A + \lambda B)}{2}\end{aligned} \tag{2.8}$$

for any $\lambda \in [0, 1]$.

If we multiply inequality (2.7) by $1-\lambda$ and (2.8) by λ, add the obtained inequalities and use representation (2.5), then we get

$$\begin{aligned}&(1-\lambda) f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right] \\ &\leq \int_0^1 f((1-t) A+t B) d t \\ &\leq(1-\lambda) \frac{f((1-\lambda) A+\lambda B)+f(B)}{2}+\lambda \frac{f(A)+f((1-\lambda) A+\lambda B)}{2},\end{aligned}$$

which proves the second and third inequalities in (2.6).

By the operator convexity of f we have

$$\begin{aligned}&(1-\lambda) f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]+\lambda f\left[\frac{(2-\lambda) A+\lambda B}{2}\right] \\ &\geq f\left[(1-\lambda) \frac{(1-\lambda) A+(1+\lambda) B}{2}+\lambda \frac{(2-\lambda) A+\lambda B}{2}\right]=f\left(\frac{A+B}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}[f((1-\lambda) A+\lambda B)+(1-\lambda) f(B)+\lambda f(A)] \\ &\leq \frac{1}{2}[(1-\lambda) f(A)+\lambda f(B)+(1-\lambda) f(B)+\lambda f(A)]=\frac{f(A)+f(B)}{2}\end{aligned}$$

that prove the first and last inequality in (2.6). □

We have:

Corollary 2. *Let $f: I \rightarrow \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned}\Phi\left(f\left(\frac{A+B}{2}\right)\right) &\leq(1-\lambda) \Phi\left(f\left[\frac{(1-\lambda) A+(1+\lambda) B}{2}\right]\right)+\lambda \Phi\left(f\left[\frac{(2-\lambda) A+\lambda B}{2}\right]\right) \\ &\leq \int_0^1 \Phi(f((1-t) A+t B)) d t \\ &\leq \frac{1}{2}[\Phi(f((1-\lambda) A+\lambda B))+(1-\lambda) \Phi(f(B))+\lambda \Phi(f(A))] \\ &\leq \frac{\Phi(f(A))+\Phi(f(B))}{2}\end{aligned} \tag{2.9}$$

for any $\lambda \in[0,1]$.

3 Bounds for HH-Difference

We consider the difference functional

$$J_n(\mathbf{p} ; \mathbf{A}, f, I):=\sum_{j=1}^n p_j f\left(A_j\right)-P_n f\left(\frac{1}{P_n} \sum_{j=1}^n p_j A_j\right) \tag{3.1}$$

where $\mathbf{p}=(p_1, \ldots, p_n)$, $p_j \geq 0$ with $j \in\{1, \ldots, n\}$ and $P_n>0$, $\mathbf{A}=(A_1, \ldots, A_n)$ is an n-tuple of selfadjoint operators with $\operatorname{Sp}\left(A_j\right) \subseteq I$ for $j \in\{1, \ldots, n\}$ and $f: I \rightarrow \mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^{+}$ the set of all n-tuples $\mathbf{p}=(p_1, \ldots, p_n)$, $p_j \geq 0$ with $j \in\{1, \ldots, n\}$ and $P_n>0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^{+}$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in\{1, \ldots, n\}$.

In [7] we established the following properties of the functional $J_n(\cdot ; \mathbf{A}, f, I)$:

Theorem 5. *Assume that $f : I \to \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1, \ldots, A_n)$ an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, I) \geq J_n(\mathbf{p}; \mathbf{A}, f, I) + J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{3.2}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p}; \mathbf{A}, f, I) \geq J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{3.3}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a monotonic functional in the operator order.

The following boundedness property also holds:

Corollary 3. *Assume that the function $f : I \to \mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1, \ldots, A_n)$ satisfies the condition $\operatorname{Sp}(A_j) \subseteq I$ for any $j \in \{1, \ldots, n\}$. If $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and there exists the positive constants m, M such that*

$$m\mathbf{q} \leq \mathbf{p} \leq M\mathbf{q}, \tag{3.4}$$

then

$$mJ_n(\mathbf{q}; \mathbf{A}, f, I) \leq J_n(\mathbf{p}; \mathbf{A}, f, I) \leq MJ_n(\mathbf{q}; \mathbf{A}, f, I) \tag{3.5}$$

in the operator order.

We observe that if all $q_j > 0$, $j \in \{1, \ldots, n\}$, then we have the inequality

$$\begin{aligned} \min_{j \in \{1,\ldots,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) &\leq J_n(\mathbf{p}; \mathbf{A}, f, I) \\ &\leq \max_{j \in \{1,\ldots,n\}} \left\{ \frac{p_j}{q_j} \right\} J_n(\mathbf{q}; \mathbf{A}, f, I) \end{aligned} \tag{3.6}$$

in the operator order.

In particular, by (3.6) for $n = 2$, $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ we get

$$\begin{aligned} &\min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) f(A) + qf(B) - f((1-q)A + qB)] \\ &\leq [(1-p) f(A) + pf(B) - f((1-p)A + pB)] \\ &\leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) f(A) + qf(B) - f((1-q)A + qB)] \end{aligned} \tag{3.7}$$

for any selfadjoint operators A and B with spectra in I.

If we take $q = \frac{1}{2}$ in (1.1), then we get

$$\begin{aligned} 2 \min\{t, 1-t\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2} \right) \right] &\leq [(1-t) f(A) + tf(B) - f((1-t)A + tB)] \\ &\leq 2 \max\{t, 1-t\} \left[\frac{f(A) + f(B)}{2} - f\left(\frac{A+B}{2} \right) \right] \end{aligned} \tag{3.8}$$

for any selfadjoint operators A and B with spectra in I and $t \in [0, 1]$.

If we take in (3.7) the map Φ, then we have

$$\begin{aligned} &\min \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) \Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))] \\ &\leq [(1-p) \Phi(f(A)) + p\Phi(f(B)) - \Phi(f((1-p)A + pB))] \\ &\leq \max \left\{ \frac{p}{q}, \frac{1-p}{1-q} \right\} [(1-q) \Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))] \end{aligned} \tag{3.9}$$

for any $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

The following result provides some upper and lower bounds for the *HH-difference*

$$\frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt.$$

Theorem 6. *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\frac{1}{2}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \le \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt \tag{3.10}$$
$$\le \frac{1}{2}\frac{q^2-q+1}{q(1-q)}[(1-q)f(A)+qf(B)-f((1-q)A+qB)]$$

for any $q \in (0,1)$.

Proof. From (3.7) we have

$$\min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)] \tag{3.11}$$
$$\le [(1-t)f(A)+tf(B)-f((1-t)A+tB)]$$
$$\le \max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)]$$

with $t \in [0,1]$ and $q \in (0,1)$.

If we integrate over $t \in [0,1]$ the inequality (3.11), then we get

$$[(1-q)f(A)+qf(B)-f((1-q)A+qB)]\int_0^1 \min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\}dt \tag{3.12}$$
$$\le \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt$$
$$\le [(1-q)f(A)+qf(B)-f((1-q)A+qB)]\int_0^1 \max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\}dt$$

for any A, B with spectra in I and $q \in (0,1)$.

Observe that

$$\frac{t}{q} - \frac{1-t}{1-q} = \frac{t-q}{q(1-q)}$$

showing that

$$\min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} = \begin{cases} \frac{t}{q} \text{ if } 0 \le t \le q \le 1 \\ \frac{1-t}{1-q} \text{ if } 0 \le q \le t \le 1 \end{cases}$$

and

$$\max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} = \begin{cases} \frac{1-t}{1-q} \text{ if } 0 \le t \le q \le 1 \\ \frac{t}{q} \text{ if } 0 \le q \le t \le 1. \end{cases}$$

Then

$$\int_0^1 \min\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\}dt = \int_0^q \frac{t}{q}dt + \int_q^1 \frac{1-t}{1-q}dt$$
$$= \frac{q^2}{2q} + \frac{1}{1-q}\left(1-q-\left(\frac{1-q^2}{2}\right)\right) = \frac{1}{2}$$

and

$$\int_0^1 \max\left\{\frac{t}{q}, \frac{1-t}{1-q}\right\} dt = \int_0^q \frac{1-t}{1-q} dt + \int_q^1 \frac{t}{q} dt$$
$$= \frac{1}{1-q}\left(q - \frac{q^2}{2}\right) + \frac{1-q^2}{2q}$$
$$= \frac{q^2-q+1}{2q(1-q)}$$

and by (3.12) we obtain the desired result (3.10). □

Corollary 4. *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\frac{1}{2}\left[(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))\right] \tag{3.13}$$
$$\leq \frac{\Phi(f(A)) + \Phi(f(B))}{2} - \int_0^1 \Phi(f((1-t)A + tB))\, dt$$
$$\leq \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[(1-q)\Phi(f(A)) + q\Phi(f(B)) - \Phi(f((1-q)A + qB))\right].$$

We also have the following bounds for the other *HH-difference*

$$\int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right).$$

Theorem 7. *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\frac{1}{2q(1-q)}\min\{1-q, q\}\left[\int_0^1 f((1-t)A + tB)\, dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A + sB)\, ds\right] \tag{3.14}$$
$$\leq \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right)$$
$$\leq \frac{1}{2q(1-q)}\max\{1-q, q\}\left[\int_0^1 f((1-t)A + tB)\, dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A + sB)\, ds\right]$$

or, equivalently

$$\frac{2q(1-q)}{\max\{1-q, q\}}\left[\int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right)\right] \tag{3.15}$$
$$\leq \int_0^1 f((1-t)A + tB)\, dt - \frac{1}{1-2q}\int_q^{1-q} f((1-s)A + sB)\, ds$$
$$\leq \frac{2q(1-q)}{\min\{1-q, q\}}\left[\int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2}\right)\right]$$

for any $q \in (0, 1)$, $q \neq \frac{1}{2}$.

Proof. If we take in (3.7) $p = \frac{1}{2}$, then we have

$$\begin{aligned}&\frac{1}{2q(1-q)}\min\{1-q,q\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)]\\&\le\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}[(1-q)f(A)+qf(B)-f((1-q)A+qB)]\end{aligned}\tag{3.16}$$

for any A, B with spectra in I and $q \in (0,1)$.

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ in (3.16), then we get

$$\begin{aligned}&\frac{1}{2q(1-q)}\min\{1-q,q\}[(1-q)f((1-t)A+tB)+qf(tA+(1-t)B)\\&-f((1-q)[(1-t)A+Bt]+q[tA+(1-t)B])]\\&\le\frac{f((1-t)A+tB)+f(tA+(1-t)B)}{2}-f\left(\frac{A+B}{2}\right)\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}[(1-q)f((1-t)A+tB)+qf(tA+(1-t)B)\\&-f((1-q)[(1-t)A+Bt]+q[tA+(1-t)B])]\end{aligned}\tag{3.17}$$

for any $A, B \in C$, $t \in [0,1]$ and $q \in (0,1)$.

If we take the integral over $t \in [0,1]$ in (3.17) and take into account that

$$\int_0^1 f((1-t)A+tB)\,dt=\int_0^1 f(tA+(1-t)B)\,dt$$

we get

$$\begin{aligned}&\frac{1}{2q(1-q)}\min\{1-q,q\}\left[\int_0^1 f((1-t)A+tB)\,dt-\int_0^1 f((1-q)[(1-t)A+tB]+q[tA+(1-t)B])\,dt\right]\\&\le\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}\left[\int_0^1 f((1-t)A+tB)\,dt-\int_0^1 f((1-q)[(1-t)A+tB]+q[tA+(1-t)B])\,dt\right]\end{aligned}\tag{3.18}$$

or any A, B with spectra in I and $q \in (0,1)$.

Observe that for any A, B with spectra in I, $t \in [0,1]$ and $q \in (0,1)$ we have

$$(1-q)[(1-t)A+tB]+q[tA+(1-t)B]=[(1-q)(1-t)+qt]A+[(1-q)t+(1-t)q]B$$

and by putting $s := (1-q)t+(1-t)q$, for $q \neq \frac{1}{2}$ we have

$$[(1-q)(1-t)+qt]A+[(1-q)t+(1-t)q]B=(1-s)A+sB.$$

If $q \neq \frac{1}{2}$, then s is a change of variable, $ds = (1-2q)\,dt$ and we have for any A, B with spectra in I that

$$\int_0^1 f((1-q)[(1-t)A+tB]+q[tA+(1-t)B])\,dt=\frac{1}{1-2q}\int_q^{1-q} f((1-s)A+sB)\,ds.$$

On making use of (3.18) we get the desired result (3.14). □

Corollary 5. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have*

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q,q\}}\left[\int_0^1 \Phi(f((1-t)A+tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right] \\ &\leq \int_0^1 \Phi(f((1-t)A+tB))\,dt - \frac{1}{1-2q}\int_q^{1-q}\Phi(f((1-s)A+sB))\,ds \\ &\leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[\int_0^1 \Phi(f((1-t)A+tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right] \end{aligned} \tag{3.19}$$

for any $q \in (0,1)$, $q \neq \frac{1}{2}$.

Remark 1. *If we take $q = \frac{1}{4}$ in (3.15) and (3.19), then we get*

$$\begin{aligned} \frac{1}{2}\left[\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right] &\leq \int_0^1 f((1-t)A+tB)\,dt - 2\int_{1/4}^{3/4} f((1-s)A+sB)\,ds \\ &\leq \frac{3}{2}\left[\int_0^1 f((1-t)A+tB)\,dt - f\left(\frac{A+B}{2}\right)\right] \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \frac{1}{2}\left[\int_0^1 \Phi(f((1-t)A+tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right] &\leq \int_0^1 \Phi(f((1-t)A+tB))\,dt - 2\int_{1/4}^{3/4} \Phi(f((1-s)A+sB))\,ds \\ &\leq \frac{3}{2}\left[\int_0^1 \Phi(f((1-t)A+tB))\,dt - \Phi\left(f\left(\frac{A+B}{2}\right)\right)\right] \end{aligned} \tag{3.21}$$

for any A, B with spectra in I and $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

4 Some Examples

The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$.

If we write the inequality (2.3) for the power $1 \leq r \leq 2$ (or $-1 \leq r \leq 0$) we have

$$\int_0^1 ((1-t)\Phi(A) + t\Phi(B))^r\,dt \leq \int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt, \tag{4.1}$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ $(A, B \in \mathcal{B}^{++}(H))$. In the case $0 \leq r \leq 1$ the inequalities reverse in (4.1).

If we write the inequality (2.9) for the power $1 \le r \le 2$ (or $-1 \le r \le 0$) we have

$$\begin{aligned}\Phi\left(\left(\frac{A+B}{2}\right)^r\right) &\le (1-\lambda)\Phi\left(\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]^r\right)+\lambda\Phi\left(\left[\frac{(2-\lambda)A+\lambda B}{2}\right]^r\right) \qquad (4.2)\\ &\le \int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt\\ &\le \frac{1}{2}\left[\Phi\left(((1-\lambda)A+\lambda B)^r\right)+(1-\lambda)\Phi\left(B^r\right)+\lambda\Phi\left(A^r\right)\right]\\ &\le \frac{\Phi\left(A^r\right)+\Phi\left(B^r\right)}{2},\end{aligned}$$

where $\lambda \in [0,1]$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ $(A, B \in \mathcal{B}^{++}(H))$. In the case $0 \le r \le 1$ the inequalities reverse in (4.2).

If we write the inequality (3.9) for the power $1 \le r \le 2$ (or $-1 \le r \le 0$) we get for $p \in [0,1]$, $q \in (0,1)$ that

$$\begin{aligned}&\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[(1-q)\Phi\left(A^r\right)+q\Phi\left(B^r\right)-\Phi\left(((1-q)A+qB)^r\right)\right] \qquad (4.3)\\ &\le \left[(1-p)\Phi\left(A^r\right)+p\Phi\left(B^r\right)-\Phi\left(((1-p)A+pB)^r\right)\right]\\ &\le \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[(1-q)\Phi\left(A^r\right)+q\Phi\left(B^r\right)-\Phi\left(((1-q)A+qB)^r\right)\right]\end{aligned}$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ $(A, B \in \mathcal{B}^{++}(H))$.

From (3.13) we have for $1 \le r \le 2$ (or $-1 \le r \le 0$) that

$$\begin{aligned}&\frac{1}{2}\left[(1-q)\Phi\left(A^r\right)+q\Phi\left(B^r\right)-\Phi\left(((1-q)A+qB)^r\right)\right] \qquad (4.4)\\ &\le \frac{\Phi\left(A^r\right)+\Phi\left(B^r\right)}{2}-\int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt\\ &\le \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[(1-q)\Phi\left(A^r\right)+q\Phi\left(B^r\right)-\Phi\left(((1-q)A+qB)^r\right)\right]\end{aligned}$$

while from(3.19) we have that

$$\begin{aligned}&\frac{2q(1-q)}{\max\{1-q,q\}}\left[\int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt-\Phi\left(\left(\frac{A+B}{2}\right)^r\right)\right] \qquad (4.5)\\ &\le \int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt-\frac{1}{1-2q}\int_q^{1-q}\Phi\left(((1-s)A+sB)^r\right)ds\\ &\le \frac{2q(1-q)}{\min\{1-q,q\}}\left[\int_0^1 \Phi\left(((1-t)A+tB)^r\right)dt-\Phi\left(\left(\frac{A+B}{2}\right)^r\right)\right],\end{aligned}$$

where $p \in [0,1]$, $q \in (0,1)$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and $A, B \in \mathcal{B}^+(H)$ $(A, B \in \mathcal{B}^{++}(H))$.

The function $f(t) = -\ln t$ is operator convex on $(0,\infty)$. Then by (2.3) we have

$$\int_0^1 \ln((1-t)\Phi(A)+t\Phi(B))\,dt \ge \int_0^1 \Phi(\ln((1-t)A+tB))\,dt \qquad (4.6)$$

while by (2.9) we have, for $\lambda \in [0, 1]$ that

$$\Phi\left(\ln\left(\frac{A+B}{2}\right)\right) \geq (1-\lambda)\,\Phi\left(\ln\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]\right) + \lambda\Phi\left(\ln\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\right) \tag{4.7}$$
$$\geq \int_0^1 \Phi(\ln((1-t)A+tB))\,dt$$
$$\geq \frac{1}{2}\left[\Phi(\ln((1-\lambda)A+\lambda B)) + (1-\lambda)\Phi(\ln(B)) + \lambda\Phi(\ln(A))\right]$$
$$\geq \frac{\Phi(\ln(A))+\Phi(\ln(B))}{2},$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^{++}(H)$.

From (3.9) we have for $p \in [0, 1]$, $q \in (0, 1)$ that

$$\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[\Phi(\ln((1-q)A+qB)) - (1-q)\Phi(\ln(A)) - q\Phi(\ln(B))\right] \tag{4.8}$$
$$\leq \left[\Phi(\ln((1-p)A+pB)) - (1-p)\Phi(\ln(A)) - p\Phi(\ln(B))\right]$$
$$\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[\Phi(\ln((1-q)A+qB)) - (1-q)\Phi(\ln(A)) - q\Phi(\ln(B))\right],$$

from (3.13) we have

$$\frac{1}{2}\left[\Phi(\ln((1-q)A+qB)) - (1-q)\Phi(\ln(A)) - q\Phi(\ln(B))\right] \tag{4.9}$$
$$\leq \int_0^1 \Phi(\ln((1-t)A+tB))\,dt - \frac{\Phi(\ln(A))+\Phi(\ln(B))}{2}$$
$$\leq \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[\Phi(\ln((1-q)A+qB)) - (1-q)\Phi(\ln(A)) - q\Phi(\ln(B))\right].$$

while from (3.19)

$$\frac{2q(1-q)}{\max\{1-q, q\}}\left[\Phi\left(\ln\left(\frac{A+B}{2}\right)\right) - \int_0^1 \Phi(\ln((1-t)A+tB))\,dt\right] \tag{4.10}$$
$$\leq \frac{1}{1-2q}\int_q^{1-q} \Phi(\ln((1-s)A+sB))\,ds - \int_0^1 \Phi(\ln((1-t)A+tB))\,dt$$
$$\leq \frac{2q(1-q)}{\min\{1-q, q\}}\left[\Phi\left(\ln\left(\frac{A+B}{2}\right)\right) - \int_0^1 \Phi(\ln((1-t)A+tB))\,dt\right]$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $A, B \in \mathcal{B}^{++}(H)$.

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