Note

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The spectrum of two interesting stochastic matrices

<https://doi.org/10.1515/spma-2020-0003>

Received September 29, 2019; accepted December 3, 2019

Abstract: The spectrum of two interesting stochastic matrices appearing in an engineering paper is completely determined. As a result, an inequality conjectured in that paper, involving two second largest eigenvalues, is easily proved.

Keywords: Stochastic matrix, Spectrum, Leader-follower model, Bipartite graph

MSC: Primary: 15B51 Secondary: 11C20

1 Introduction

Two interesting matrices, **A** and **B** below, appear in the paper [1], by Roya Norouzi Kandalan et al. The paper investigates the leader-follower model in decision making, by using matrix analysis techniques associated to bipartite graphs. The two $(n \times n)$ matrices of interest are right stochastic with rational entries. Therefore, the

vector $\mathbf{u}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$, $\mathbb{R}$ the reals, is an eigenvector with (largest) eigenvalue $\lambda_1 = 1$ for both matrices. It is

inferred that the iterative model proposed converges faster because

$$\lambda_2(\mathbf{BA}) \leq \lambda_2(\mathbf{A}^2), \quad (1)$$

where λ_2 is the second largest eigenvalue of the respective matrices. In this note we validate the inequality (1) via a complete determination of the spectra of **A** and **B**. Our approach is elementary, and is successful due to the large $((n-2)$ -dimensional) null-spaces exhibited by the two matrices.

2 The Matrices

To the end of introducing the two matrices **A** and **B** it is more economical to use block-matrix notation. Fix first two positive integers, n and k , $1 < k < n$. If p is an arbitrary positive integer, denote by $\mathbf{U}_p \in \mathbb{R}^{p \times p}$ the

square $p \times p$ -matrix with all entries equal to 1, i. e., $\mathbf{U}_p = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}$, and by $\mathbf{u}_p$ the vector in $\mathbb{R}^p$ with all

components 1, similar to $\mathbf{u}_n$ above. Also, denote by $\mathbf{O}_{p,q} \in \mathbb{R}^{p \times q}$ the zero-matrix with p rows and q columns, and by superscript $\cdot^T$ transposition of matrices or of vectors viewed as matrices. In particular, $\mathbf{O}_{p,1}$ becomes the vector $\mathbf{o}_p$ in $\mathbb{R}^p$.

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Then define in block-matrix notation matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^{n \times n}$ by

$$\mathbf{A} = \begin{bmatrix} \frac{1}{k} \mathbf{U}_{k-1} & \frac{1}{k} \mathbf{u}_{k-1} & \mathbf{0}_{k-1, n-k} \\ \frac{1}{2k} \mathbf{u}_{k-1}^T & \frac{n+1}{2k(n-k+1)} & \frac{1}{2(n-k+1)} \mathbf{u}_{n-k}^T \\ \mathbf{0}_{n-k, k-1} & \frac{1}{n-k+1} \mathbf{u}_{n-k} & \frac{1}{n-k+1} \mathbf{U}_{n-k} \end{bmatrix} \quad (2)$$

and

$$\mathbf{B} = \begin{bmatrix} \frac{1}{n-k+1} & \mathbf{0}_{k-2}^T & \frac{1}{n-k+1} & 0 & \frac{1}{n-k+1} \mathbf{u}_{n-k-1}^T \\ \mathbf{0}_{k-2} & \frac{1}{k} \mathbf{U}_{k-2} & \frac{1}{k} \mathbf{u}_{k-2} & \frac{1}{k} \mathbf{u}_{k-2} & \mathbf{0}_{k-2, n-k-1} \\ \frac{1}{2(n-k+1)} & \frac{1}{2k} \mathbf{u}_{k-2}^T & \frac{n+1}{2k(n-k+1)} & \frac{1}{2k} & \frac{1}{2(n-k+1)} \mathbf{u}_{n-k-1}^T \\ 0 & \frac{1}{k} \mathbf{u}_{k-2}^T & \frac{1}{k} & \frac{1}{k} & \mathbf{0}_{n-k-1}^T \\ \frac{1}{n-k+1} \mathbf{u}_{n-k-1} & \mathbf{0}_{n-k-1, k-2} & \frac{1}{n-k+1} \mathbf{u}_{n-k-1} & \mathbf{0}_{n-k-1} & \frac{1}{n-k+1} \mathbf{U}_{n-k-1} \end{bmatrix} \quad (3)$$

3 The Results

In preparation for stating our key lemma and main result we denote the standard basis of $\mathbb{R}^n$ by $\mathbf{e}_n^1, \mathbf{e}_n^2, \dots$,

$\mathbf{e}_n^n$, where $\mathbf{e}_n^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_n^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$, $\dots$, $\mathbf{e}_n^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. We also denote a generic vector in $\mathbb{R}^n$ by $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Lastly, $\mathbb{R}\mathbf{v}$

means the 1-dimensional subspace of $\mathbb{R}^n$ spanned by the vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}_n$.

Lemma. For fixed $1 < k < n$, both matrices $\mathbf{A}$ and $\mathbf{B}$ are diagonalizable, with the same spectrum, including multiplicities: $\lambda_1(\mathbf{A}) = \lambda_1(\mathbf{B}) = 1$ with multiplicity 1 (and eigenvector $\mathbf{u}_n$), $\lambda_2(\mathbf{A}) = \lambda_2(\mathbf{B}) = 1 - \frac{1}{2k} - \frac{1}{2(n-k+1)}$ with multiplicity 1, and 0, with multiplicity $n-2$.

a) The null-space of $\mathbf{A}$ is $\mathcal{N}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^k x_i = 0, \sum_{i=k}^n x_i = 0 \right\}$. An eigenvector for $\lambda_2(\mathbf{A})$ is $\mathbf{v}_\mathbf{A} = -\frac{1}{n+1} \mathbf{u}_n + \mathbf{e}_n^1 + \frac{1}{\lambda_2(\mathbf{A})} \mathbf{y}$, where $\mathbf{y} \in \mathcal{N}(\mathbf{A})$ and has components

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad y_i = \begin{cases} \frac{2(k-1)^2 - n(2k-3)}{2k(n-k+1)}, & i = 1 \\ \frac{2(n-k)+1}{2k(n-k+1)}, & 2 \leq i \leq k-1 \\ \frac{n-k}{2k(n-k+1)}, & i = k \\ -\frac{1}{2k(n-k+1)}, & k+1 \leq i \leq n \end{cases} \quad (4)$$

b) The null-space of $\mathbf{B}$ is $\mathcal{N}(\mathbf{B}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=2}^{k+1} x_i = 0, x_1 + x_k + \sum_{i=k+2}^n x_i = 0 \right\}$. An eigenvector for $\lambda_2(\mathbf{B})$ is $\mathbf{v}_B = -\frac{1}{n+1}\mathbf{u}_n + \mathbf{e}_n^1 + \frac{1}{\lambda_2(\mathbf{B})}\mathbf{z}$, where $\mathbf{z} \in \mathcal{N}(\mathbf{B})$ and has components

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad z_i = \begin{cases} \frac{n-2k(n-k)}{2k(n-k+1)}, & i=1 \\ -\frac{1}{2k(n-k+1)}, & 2 \leq i \leq k-1 \text{ or } i=k+1 \\ \frac{k-1}{2k(n-k+1)}, & i=k \\ \frac{2k-1}{2k(n-k+1)}, & k+2 \leq i \leq n \end{cases} \quad (5)$$

Proof. a) That $\mathcal{N}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{i=1}^k x_i = 0, \sum_{i=k}^n x_i = 0 \right\}$ is a routine verification, based on the actual expression of $\mathbf{A}$. Therefore, $\dim_{\mathbb{R}} \mathcal{N}(\mathbf{A}) = n-2$, and so we have that 0 is an eigenvalue of $\mathbf{A}$ with geometric multiplicity $n-2$. As a stochastic matrix $\mathbf{A}$ also possesses 1 as a (necessarily largest) eigenvalue, for which $\mathbf{u}_n$ is an eigenvector. Notice now that we have the direct sum decomposition

$$\mathbb{R}^n = \mathbb{R}\mathbf{u}_n \oplus \mathbb{R}\mathbf{e}_n^1 \oplus \mathcal{N}(\mathbf{A}).$$

We seek the last eigenvector of $\mathbf{A}$ to be of the form $\mathbf{v} = \alpha\mathbf{u}_n + \mathbf{e}_n^1 + \mathbf{x}$, for suitable $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{N}(\mathbf{A})$. Equivalently, for some $\lambda \in \mathbb{R}$ we must have $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, or

$$\alpha\mathbf{u}_n + \begin{bmatrix} \frac{1}{k}\mathbf{u}_{k-1} \\ \frac{1}{2k} \\ \mathbf{o}_{n-k} \end{bmatrix} = \lambda\alpha\mathbf{u}_n + \lambda\mathbf{e}_n^1 + \lambda\mathbf{x}. \quad (6)$$

From the above equation and $\lambda\mathbf{x} \in \mathcal{N}(\mathbf{A})$ we infer, via the description of $\mathcal{N}(\mathbf{A})$, that

$$k\alpha(1-\lambda) + 1 - \frac{1}{2k} - \lambda = 0, \text{ and } (n-k+1)\alpha(1-\lambda) + \frac{1}{2k} = 0.$$

Eliminating first $\alpha(1-\lambda)$ between these two equations yields

$$\lambda = 1 - \frac{1}{2k} - \frac{1}{2(n-k+1)}, \text{ and further } \alpha = -\frac{1}{n+1}.$$

Setting now $\mathbf{y} = \lambda\mathbf{x}$, the components of $\mathbf{y}$ as given by Equations (4) follow immediately from (6) and the expressions of α and λ in terms of n and k . Notice that $0 < \lambda < 1$, therefore λ is indeed the second largest eigenvalue of $\mathbf{A}$.

b) The proof of b) can mimic exactly that of a), and we encourage the reader to do it. However, there is an easy way to conclude b) from a), via the observation that the matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, with similarity matrix $\mathbf{P}$ in column form given by

$$\mathbf{P} = [\mathbf{e}_n^{k+1}, \mathbf{e}_n^2, \dots, \mathbf{e}_n^k, \mathbf{e}_n^1, \mathbf{e}_n^{k+2}, \dots, \mathbf{e}_n^n]$$

$\mathbf{P}$ is a nontrivial involution, $\mathbf{P}^2 = \mathbf{Id}$, and so $\mathbf{P}^{-1} = \mathbf{P}$. Right multiplication by $\mathbf{P}$ of any matrix swaps the first and the $(k+1)$ th columns of the matrix, while leaving the other columns unchanged, and left multiplication does similar things to the rows. This being said, notice that indeed $\mathbf{A}$ is obtained from $\mathbf{B}$ by first swapping columns 1 and $k+1$, and then swapping rows 1 and $k+1$ of the resulting matrix. Thus, $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}$, or $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, and as a result $\mathbf{A}$ and $\mathbf{B}$ have the same spectrum, and $\mathcal{N}(\mathbf{B}) = \mathcal{N}(\mathbf{A}\mathbf{P})$. b) then follows. Indeed, similarity gives that $\mathbf{P}\mathbf{v}_A$ is an eigenvector of $\mathbf{B}$ with eigenvalue $\lambda_2(\mathbf{B}) = \lambda_2(\mathbf{A})$, however $\mathbf{P}\mathbf{v}_A \neq \mathbf{v}_B$. Notice that

$$\mathbf{P}\mathbf{v}_A = -\frac{1}{n+1}\mathbf{u}_n + \mathbf{e}_n^{k+1} + \frac{1}{\lambda_2(\mathbf{A})}\mathbf{P}\mathbf{y},$$

and $\mathbf{P}\mathbf{v}_\mathbf{A} = \mathbf{v}_\mathbf{B}$ would imply $\mathbf{e}_n^{k+1} - \mathbf{e}_n^1 \in \mathcal{N}(\mathbf{B})$, which is not true. However, since $\lambda_2(\mathbf{B})$ is an eigenvalue of $\mathbf{B}$ with geometric multiplicity 1, $\mathbf{P}\mathbf{v}_\mathbf{A}$ and $\mathbf{v}_\mathbf{B}$ must be proportional. The proportionality constant equals eventually $-\frac{k}{n-k+1}$, from which the form of $\mathbf{v}_\mathbf{B}$ follows from that of $\mathbf{P}\mathbf{v}_\mathbf{A}$. $\square$

Theorem. For positive integers $1 < k < n$ and stochastic matrices $\mathbf{A}$ and $\mathbf{B}$ defined by Equations (2) and (3) above, we have

$$\lambda_2(\mathbf{BA}) \leq \lambda_2(\mathbf{A}^2),$$

where λ_2 denotes the second largest eigenvalue of the respective matrices.

Proof. Clearly, $\mathbf{A}^2$ is a diagonalizable matrix, and by the Lemma it has second largest eigenvalue

$$\lambda_2(\mathbf{A}^2) = \left(1 - \frac{1}{2k} - \frac{1}{2(n-k+1)}\right)^2.$$

For $\lambda_2(\mathbf{BA})$ we have to work a little more. Again by the Lemma, $\mathbf{BA} = (\mathbf{PAP})\mathbf{A} = (\mathbf{PA})^2$, so $\lambda_2(\mathbf{BA})$ can be read off from the spectrum of $\mathbf{PA}$. As $(\mathbf{PA})\mathbf{u}_n = \mathbf{u}_n$ and $\mathcal{N}(\mathbf{PA}) = \mathcal{N}(\mathbf{A})$, we have to investigate if $\mathbf{PA}$ has one more eigenvalue, much like in the proof of the Lemma, part a). We seek an eigenvector of $\mathbf{PA}$ of the form $\alpha\mathbf{u}_n + \mathbf{e}_n^1 + \mathbf{x}$ for some scalar α and $\mathbf{x} \in \mathcal{N}(\mathbf{A})$, and some eigenvalue $\lambda \neq 0$. Then

$$\alpha\mathbf{u}_n + \mathbf{P}\mathbf{a}_n^1 = \lambda\alpha\mathbf{u}_n + \lambda\mathbf{e}_n^1 + \lambda\mathbf{x},$$

where $\mathbf{a}_n^1$ is the first column of $\mathbf{A}$. Equivalently,

$$\alpha(1-\lambda)\mathbf{u}_n + \begin{bmatrix} 0 \\ \frac{1}{k}\mathbf{u}_{k-2} \\ \frac{1}{2k} \\ \frac{1}{k} \\ \mathbf{0}_{n-k-1} \end{bmatrix} - \lambda\mathbf{e}_n^1 \in \mathcal{N}(\mathbf{A}),$$

or

$$k\alpha(1-\lambda) + 1 - \frac{3}{2k} - \lambda = 0, \text{ and } (n-k-1)\alpha(1-\lambda) + \frac{3}{2k} = 0.$$

We conclude that

$$\lambda = 1 - \frac{3}{2k} - \frac{3}{2(n-k+1)}, \text{ and } \alpha = -\frac{1}{n+1}.$$

Unlike before this λ may be non-positive, but at any rate

$$\lambda_2(\mathbf{BA}) = \left(1 - \frac{3}{2k} - \frac{3}{2(n-k+1)}\right)^2.$$

Therefore, $\lambda_2(\mathbf{BA}) \leq \lambda_2(\mathbf{A}^2)$ if and only if

$$\left|1 - \frac{3}{2k} - \frac{3}{2(n-k+1)}\right| \leq 1 - \frac{1}{2k} - \frac{1}{2(n-k+1)},$$

an inequality that is easily seen to hold true for every $1 < k < n$.

When $1 - \frac{3}{2k} - \frac{3}{2(n-k+1)} = 0$ no eigenvector of the form $\alpha\mathbf{u}_n + \mathbf{e}_n^1 + \mathbf{x}$ exists for $\mathbf{PA}$, so $\mathbf{PA}$ is not diagonalizable. This can happen only when

$$n = k + \frac{k+3}{2k-3}, \text{ i.e., } (k=2, n=7), (k=3, n=5), \text{ and } (k=6, n=7).$$

However, $\mathbf{BA}$ continues to be diagonalizable in those cases too, and the eigenvalue 0 has geometric multiplicity $n-1$. $\square$

References

- [1] R. Norouzi Kandalan, R. Sing, K. Namuduri, M. Varanasi, B. Buckles *Impact of Mobility on the Collective Decision Making in the Leader-Follower Model*, preprint