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Some Characterizations of the Distribution of the Condition Number of a Complex Gaussian Matrix

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Abstract: The objective of this paper is to characterize the distribution of the condition number of a complex Gaussian matrix. Several new distributional properties of the distribution of the condition number of a complex Gaussian matrix are given. Based on such distributional properties, some characterizations of the distribution are given by truncated moment, order statistics and upper record values.

Keywords: Characterization, Condition number, Gaussian matrices, Order statistics, Truncated first moment, Upper record values

MSC: 15A12, 15A52

1 Introduction

Gaussian matrices appear in many fields of pure and applied sciences, such as, probability, statistics, multivariate statistics, linear algebra, actuarial science, physics, wireless communications, and polarimetric synthetic aperture radar (PolSAR). Many researchers have studied the distributions of the condition numbers of Gaussian and Wishart matrices, their properties and applications, for which the interested readers are referred to Demmel (1988), Edelman (1988, 1992), Chen and Dongarra (2005), Anderson and Wells (2009), Wei et al. (2011), Zhong et al. (2011), Zhang et al. (2016), and Deng et al. (2017), among others. For some related research papers, the interested readers are also referred to Lei et al. (2014), and references therein.

Edelman (1992) studied the distribution of Demmel condition number of complex Wishart matrices. Wei et al. (2011) derived the probability density function (PDF) of Demmel condition number (DCN) for complex Wishart matrices using Mellin transformation. Shakil and Ahsanullah (2016) studied the characterizations of the distribution of the Demmel condition number of real Wishart matrices. Recently, Shakil and Ahsanullah (2017) studied the characterizations of the distribution of the Demmel condition number of complex Wishart matrices. Also, Shakil and Ahsanullah (2018) studied the characterizations of the distribution of the condition numbers of real Gaussian matrices. Shi et al. (2013) studied the exact distribution of the condition number of complex random matrices. In this paper, we study some distributional properties and characterizations of the distribution of the condition numbers of complex Gaussian matrices.

The organization of this paper is as follows. In Section 2, some preliminaries are given. In Section 3, several distributional properties of the distributions of the condition numbers of complex Gaussian matrices are presented. The characterizations of the distributions of the condition numbers of complex Gaussian matrices are provided in Section 4. In Section 5, we have computed the percentage points. In Section 6, the concluding remarks are given.

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2 Some Preliminaries

In this section, we will first provide some preliminaries on the notions of the condition number of a matrix which will be relevant for establishing the various results of our paper. Then, a brief discussion of complex Gaussian matrix will be presented.

2.1 Condition Number of a Matrix

Condition number of a matrix is one of the widely used matrix features and has significant applications in many areas of numerical analysis, linear algebra, statistics, physics and engineering. The idea of the condition number of a matrix was first introduced in 1948 by Turing (1948). Further developments continued with the contributions of Rice (1966), Skeel (1979), Geurts (1982), Demmel (1987), Edelman (1988), Trefethen and Bau (1997), Bottcher and Grudsky (1998), Trefethen and Viswanath (1998), and Higham (2002), among others. For recent work on condition numbers, see, for example, Hargreaves (2004), Xu and Zhang (2004), Acosta et al. (2006), and Ern and Guermond (2006), and references therein.

2.1.1 Definition

For a given non-singular matrix $A \in \mathbb{R}^{n \times n}$, and a matrix norm $\|\cdot\|$, a non-negative quantity $\kappa(A)$ given by

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|$$

is defined as the condition number of A and is denoted by $\text{Cond}(A)$, see, for example, Datta (1995), and Leon (2006), among others. Since the condition number is defined in terms of a particular matrix norm, many different matrix norms may be chosen. The most frequently used norm is the p-norms denoted as $\|A\|_p$, ($p \geq 1$). Some commonly used p-norms are maximum column-sum norm $= \|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |a_{ij}|$, where a_{ij} 's denote the components of the matrix A , maximum row-sum norm $= \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$, and the Hilbert or spectral norm defined as $\|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^T A}$, where A^T denotes the transpose of the matrix A , and if A is Hermitian or real and symmetric, then $\|A\|_2 = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A defined as $\rho(A) = \max_i |\lambda_i|$, where λ_i 's are the eigenvalues of A . The condition number corresponding to the p-norm will be denoted $\kappa_p(A)$. For a singular square matrix $A^{n \times n}$, that is, if A is not invertible, the condition number of A is infinite, that is, $\kappa(A) = +\infty$.

2.1.2 Condition Number Based on Spectral Norm

Let $\|\cdot\|$ denote the spectral norm. Let A be a nonsingular $n \times n$ matrix. Let $\sigma_1, \sigma_2, \dots, \sigma_n$ denote the n singular values of A .

Definition 1: Let $\lambda_1, \lambda_2, \dots, \lambda_n$ denote the corresponding n eigenvalues of A . Then the condition number $\kappa_2(A)$ of A is defined by

$$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \frac{\max_{1 \leq i \leq n} |\sigma_i|}{\min_{1 \leq i \leq n} |\sigma_i|},$$

where $\max_{1 \leq i \leq n} |\sigma_i|$ and $\min_{1 \leq i \leq n} |\sigma_i|$ denote the largest and smallest singular values of A respectively.

Definition 2. Let $\cdot$ denote the spectral norm. Let A be a nonsingular $n \times n$ matrix. If A is Hermitian or real and symmetric, then the condition number $\kappa_2(A)$ of A is defined by

$$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \frac{\max_{1 \leq i \leq n} |\lambda_i|}{\min_{1 \leq i \leq n} |\lambda_i|},$$

where $\max_{1 \leq i \leq n} |\lambda_i|$ and $\min_{1 \leq i \leq n} |\lambda_i|$ are, respectively, the largest and smallest eigenvalues in modulus of A .

Note that if the columns of A are orthogonal, the condition number, $\kappa_2(A)$, of A attains its minimum bound, that is, $\kappa_2(A) \geq 1$, (see, for example, Golub and Van Loan (1996), among others). If $\kappa_2(A)$ is large, then A is said to be an ill-conditioned matrix. If $\kappa_2(A)$ is near unity, then A is said to be a well-conditioned matrix. Matrices with small condition numbers are said to be well-conditioned.

2.2 Complex Gaussian Matrices

A matrix is called a random matrix if the entries of the matrix are random numbers from a specified distribution. If the distribution is Gaussian, then we call it a Gaussian random matrix. Consider an $m \times n$ rectangular random matrix X all of whose elements are independent standard normal random variables. We call such a random matrix (or its distribution) as a Gaussian matrix, and denote it by $G(m, n)$.

A complex standard normally distributed random variable can be defined as $u + vi$, $i = \sqrt{-1}$, where u and v are independent and identically distributed (i.i.d.) standard normal random variables. If all the components of a random matrix are independent complex standard normal random variables, we call it a complex Gaussian matrix; see, for example, Haagerup, and Thorbjørnsen (2003). Let $\tilde{X}$ be such a complex Gaussian matrix of the form $X_1 + iX_2$, where X_1 and X_2 are independent and have the distribution $G(m, n)$. Let $\tilde{G}(m, n)$ denote the distribution of $\tilde{X}$. Let $\tilde{X}^T$ denote the transpose of the $n \times n$ random matrix $\tilde{X}$. Then, the $n \times n$ random matrix $W_{n \times n} = \tilde{X}\tilde{X}^T$ is said to be a Wishart matrix or to have the Wishart distribution with parameters (n, n) . That is, $W_{n \times n} (= \tilde{X}\tilde{X}^T) \sim W(n, n)$.

Further, let $\lambda_{\max} = \lambda_1 > \lambda_2 > \dots > \lambda_n = \lambda_{\min} > 0$ and $\sigma_1 > \sigma_2 > \dots > \sigma_n > 0$ denote the distinct eigenvalues of the matrix $W_{n \times n} = \tilde{X}\tilde{X}^T$ and singular values of $\tilde{X}$, respectively. Note that the squares of the singular values of $\tilde{X}$ are the eigenvalues of the Wishart matrix $W_{n \times n} = \tilde{X}\tilde{X}^T$. Then, the random quantity $\kappa_2(\tilde{X}) = \sqrt{\frac{\lambda_1}{\lambda_n}} = \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \frac{\sigma_1}{\sigma_n}$, the square root of the ratio of largest to smallest eigenvalues of the Wishart matrix $W_{n \times n} = \tilde{X}\tilde{X}^T$, is defined as the 2-norm (or the standard) condition number of $\tilde{X}$, see, for example, Edelman (1988).

3 Distributional Properties of the Condition Numbers of Complex Gaussian Matrices

In this section, we discuss the properties of the distribution of the condition numbers of complex Gaussian matrices. For the sake of simplicity and without loss of generality, we shall consider the explicit expressions of the probability density function (pdf) of the exact distribution of the distribution of the condition number, $\tilde{\kappa}$, of a complex Gaussian matrix, $\tilde{X}$, having the distribution $\tilde{G}(2, n)$.

3.1 Probability Density Function

Edelman (1988) introduced a distribution for a continuous random variable X to study the distribution $\tilde{G}(2, n)$ of the condition number, $\tilde{\kappa}$, of a complex Gaussian matrix, $\tilde{X}$, with the probability density function (pdf), $f(x)$, given by

$$f(x) = \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}}, \quad x > 1, n \geq 2. \quad (3.1)$$

By using the substitution $t^2 + 1 = \frac{1}{u}$, the corresponding cumulative distribution function (cdf) is easily derived as follows:

$$\begin{aligned} F(x) &= \int_1^x \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{t^{2n-3}(t^2-1)^2}{(t^2+1)^{2n}} dt \\ &= 1 - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} [B_{1/(1+x^2)}(n-1, n-1) - 4B_{1/(1+x^2)}(n, n)], \end{aligned} \quad (3.2)$$

where $B_x(a, b) = \int_0^x u^{a-1}(1-u)^{b-1} du$ denotes the incomplete beta function, see, for example, Abramowitz and Stegun (1970), and Gradshteyn and Ryzhik (1990). For some selected values of the parameter, n , the graphs of $f(x)$ and $F(x)$ are given in Figures 1 and 2 respectively. The effects of the parameters are obvious from these figures, that is, the distribution appears to be unimodal and right skewed.

Figure 1: Plots of the pdf (3.3) for $n = 2, 3, 4, 5$

3.2 Survival and Hazard Functions

The survival and hazard functions are respectively given by

$$S(x) = 1 - F_X(x) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} [B_{1/(1+x^2)}(n-1, n-1) - 4B_{1/(1+x^2)}(n, n)], \quad (3.3)$$

and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{2x^{2n-3}(x^2-1)^2}{[B_{1/(1+x^2)}(n-1, n-1) - 4B_{1/(1+x^2)}(n, n)](x^2+1)^{2n}} \quad (3.4)$$

where $x > 1$, $n \geq 2$. The possible shapes of $h(x)$ are provided for some selected values of the parameter, n , in the Figure 3. The effects of the parameters can easily be seen from these graphs. The upside down bathtub shape behaviors of $h(x)$ are evident from the figure.

Figure 2: Plots of the cdf (3.2) for $n = 2, 3, 4, 5$

3.3 Moment

The j th moment, $E(X^j)$, where $j > 0$ is an integer is given by

$$E(X^j) = \int_1^{\infty} x^j f(x) dx = \int_1^{\infty} \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n+j-3}(x^2-1)^2}{(x^2+1)^{2n}} dx,$$

from which on substituting $x^2 + 1 = \frac{1}{u}$, we obtain

$$\begin{aligned} E(X^j) &= \int_0^{1/2} \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} u^{2n} \left(\frac{1-u}{u} \right)^{n-2+\frac{j}{2}} \left(\frac{1-2u}{u} \right)^2 \frac{1}{u^2} du \\ &= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \int_0^{1/2} u^{n-2-\frac{j}{2}} (1-u)^{n-2+\frac{j}{2}} [1-4u(1-u)] du \\ &= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \left[B_{1/2} \left(n - \frac{j}{2} - 1, n + \frac{j}{2} - 1 \right) - 4B_{1/2} \left(n - \frac{j}{2}, n + \frac{j}{2} \right) \right], \end{aligned} \quad (3.5)$$

since $B_x(a, b) = \int_0^x u^{a-1}(1-u)^{b-1} du$. When $j = 1$ in Eq. (3.5), the first moment is given by

$$E(X) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \left[B_{1/2} \left(n - \frac{3}{2}, n - \frac{1}{2} \right) - 4B_{1/2} \left(n - \frac{1}{2}, n + \frac{1}{2} \right) \right], \quad (3.6)$$

For some values of the parameter, $n \geq 2$, the first moment, $E(X)$, is tabulated from Eq. (3.6), in Table 1.

3.4 Entropy

The entropy measure of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics, among others. According to Shannon (1948), the

Figure 3: Plots of the hazard rate function, $h(x)$, for $n = 2, 3, 4, 5$

Table 1: Table for the first moment, $E(X)$

n	$E(X)$
2	4.3562
3	2.4726
4	2.0385
5	1.8330
6	1.7096
7	1.6258
8	1.5646
9	1.5174
10	1.4798

entropy measure of a continuous real random variable X is defined as

$$H_X[f_X(X)] = E[-\ln(f_X(X))] = - \int_{-\infty}^{\infty} f_X(x) \ln[f_X(x)] dx.$$

Thus, the entropy measure of the random variable X having the pdf $f(x)$ as in Eq. (3.1) is given by

$$\begin{aligned} H_X[f_X(X)] &= - \int_1^{\infty} \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}} \ln \left[\frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}} \right] dx \\ &= 2nE(\ln(x^2+1)) - 2E(\ln(x^2-1)) - (2n-3)E(\ln(x)) - \ln \left(\frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \right), \end{aligned}$$

where $n \geq 2$, and $x > 1$. It is obvious from the above that we cannot obtain a closed form expression for the Shannon entropy. For some values of the parameter $n \geq 2$, using the Maple software, the Shannon entropy is tabulated in Table 2, and the corresponding graph is sketched in Figure 4 below. From Table 2 and Figure 4, it is observed that the Shannon entropy is a monotonic decreasing and concave up function of n .

Table 2: Table for the Shannon entropy

n	$H_X[f_X(X)] = E[-\ln(f_X(X))]$
2	2.0416
3	1.2307
4	0.8380
5	0.5853
6	0.4012
7	0.2574
8	0.1399
9	0.0411
10	-0.0442

Figure 4: Plot of the Shannon entropy, $H_X[f_X(X)] = E[-\ln(f_X(X))]$, for $n \geq 2$

4 Characterization Results

In this section, we give our proposed characterizations of the distribution of condition number of a complex Gaussian matrix by truncated moment, order statistics and upper record values.

Many authors and researchers have studied the characterizations of probability distributions, among them Galambos and Kotz (1978), Kotz and Shanbhag (1980), and Nagaraja (2006), are notable. Since the truncated distributions arise in practical statistics where the ability of record observations is limited to a given threshold or within a specified range, there has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments, that is, by considering a product of reverse hazard rate and another function of the truncated point, see, for example, Ahsanullah et al. (2015, 2016), and Ahsanullah and Shakil (2018).

Recently, many researchers have also used the idea of truncated moment method in the studies of the characterizations of the distributions of the condition numbers (DCN) of random matrices, see, for example, Shakil and Ahsanullah (2016, 2017, 2018). It appears from the literature that no attention has been paid to the characterizations of the distribution of the condition number of a complex Gaussian matrix. Therefore, motivated by the importance of the characterizations of probability distributions in practical problems, in this section we provide some new characterization results of the distribution of the condition

number, $\tilde{\kappa}$, of a complex Gaussian matrix, $\tilde{X}$, having the distribution $\tilde{G}(2, n)$, with pdf (3.1) and cdf (3.2), by truncated moment, order statistics and upper record values.

For this, we will need the following assumption and lemmas. We assume that $\frac{df(x)}{dx}$ exists for all x , where $1 < x < \infty$. We also assume that $E(X)$ exists, which is true in our case, since the pdf (3.1) has finite $E(X)$ and, therefore, can easily be characterized using conditional expectation of the first moment.

Assumption 4.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\gamma = \inf\{x|F(x) > 0\}$, and $\delta = \sup\{x|F(x) < 1\}$, $f(x)$ is a differentiable function of x in the interval (γ, δ) , and $E(X)$ exists.

Lemma 4.1. Under the Assumption 4.1, if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_{\gamma}^x \frac{u-g'(u)}{g(u)} du$ is finite for $x > \gamma$, then $f(x) = ce^{\int \frac{x-g'(x)}{g(x)} dx}$, where c is a constant determined by the condition $\int_{\gamma}^{\delta} f(x)dx = 1$.

Proof. For proof, see Shakil and Ahsanullah (2018). $\square$

Lemma 4.2. Under the Assumption 4.1, if $E(X|X \geq x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $h(x)$ is a continuous differentiable function of x with the condition that $\int_x^{\delta} \frac{u+h'(u)}{h(u)} du$ is finite for $x < \delta$, then $f(x) = ce^{\int -(\frac{h(x)+h'(x)}{h(x)})dx}$, where c is a constant determined by the condition $\int_{\gamma}^{\delta} f(x)dx = 1$.

Proof. For proof, see Shakil and Ahsanullah (2018). $\square$

4.1 Characterization by Truncated Moment

Theorem 4.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $1 = \inf\{x|F(x) > 0\}$, $\infty = \sup\{x|F(x) < 1\}$, and $E(X)$ is finite for all x , where $1 < x < \infty$. Then, $E(X|X \leq x) = g(x)\frac{f(x)}{F(x)}$, where

$$g(x) = \frac{(x^2 + 1)^{2n}P(x)}{2x^{2n-3}(x^2 - 1)^2}, \text{ and}$$

$$P(x) = \left\{ \left[B_{1/2}(n - \frac{3}{2}, n + \frac{1}{2}) - 4B_{1/2}(n - \frac{1}{2}, n + \frac{3}{2}) \right] - \left[B_{1/(1+x^2)}(n - \frac{3}{2}, n + \frac{1}{2}) - 4B_{1/(1+x^2)}(n - \frac{1}{2}, n + \frac{3}{2}) \right] \right\},$$

if and only if

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, x > 1, n \geq 2.$$

Proof. Necessary Part: Consider the pdf (3.1), that is,

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, x > 1, n \geq 2.$$

Then, since $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$, and also $E(X|X \leq x) = \frac{\int_1^x uf(u)du}{F(x)}$, we have

$$f(x)g(x) = \int_1^x tf(t)dt = \int_1^x 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{t^{2n-2}(t^2 - 1)^2}{(t^2 + 1)^{2n}} dt. \quad (4.1)$$

Let $t^2 + 1 = \frac{1}{u}$ in Eq. (4.1). Then, proceeding in the same manner as in Eq. (3.5) of sub-section 3.3, after simplification, we have

$$f(x)g(x) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \left\{ \left[B_{1/2}(n - \frac{3}{2}, n - \frac{1}{2}) - 4B_{1/2}(n - \frac{1}{2}, n + \frac{1}{2}) \right] - \left[B_{1/(1+x^2)}(n - \frac{3}{2}, n - \frac{1}{2}) - 4B_{1/(1+x^2)}(n - \frac{1}{2}, n + \frac{1}{2}) \right] \right\},$$

$$= \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} P(x), \quad (4.2)$$

where

$$P(x) = \left\{ \left[B_{1/2}(n - \frac{3}{2}, n - \frac{1}{2}) - 4B_{1/2}(n - \frac{1}{2}, n + \frac{1}{2}) \right] - \left[B_{1/(1+x^2)}(n - \frac{3}{2}, n - \frac{1}{2}) - 4B_{1/(1+x^2)}(n - \frac{1}{2}, n + \frac{1}{2}) \right] \right\}. \quad (4.3)$$

Thus, we have

$$g(x) = \frac{(x^2 + 1)^{2n} P(x)}{2x^{2n-3}(x^2 - 1)^2}.$$

This completes the proof of the necessary part.

Sufficient Part: Conversely, suppose that

$$g(x) = \frac{P(x)(x^2 + 1)^{2n}}{2x^{2n-3}(x^2 - 1)^2},$$

where $P(x)$ is given by Eq. (4.3). Then, differentiating both sides of the above equation with respect to x , and noting that $\frac{d(B_x(a, b))}{dx} = (1 - u)^{b-1} u^{a-1}$, we obtain

$$\begin{aligned} g'(x) &= x - \frac{P(x)(x^2 + 1)^{2n}}{2x^{2n-3}(x^2 - 1)^2} \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right) \\ &= x - g(x) \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right), \end{aligned}$$

from which, we get

$$\frac{x - g'(x)}{g(x)} = \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right).$$

Since, by Lemma 4.1, we have $\frac{x - g'(x)}{g(x)} = \frac{f'(x)}{f(x)}$, therefore, we have

$$\frac{f'(x)}{f(x)} = \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right).$$

On integrating both sides of the above equation with respect to x , we obtain

$$\ln f(x) = \ln(c) + (2n-3) \ln(x) + 2 \ln(x^2 - 1) - 2n \ln(x^2 + 1),$$

or,

$$f(x) = \frac{cx^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, \quad (4.4)$$

where c is a constant to be determined. Now, integrating the Eq. (4.4) with respect to x from 1 to ∞ , using the boundary condition $\int_1^\infty f(x) dx = 1$, we obtain $c = \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}$. Then, we have

$$f(x) = \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, \quad x > 1, n \geq 2.$$

This completes the proof of Theorem 4.1. $\square$

Theorem 4.2. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $1 = \inf\{x | F(x) > 0\}$, $\infty = \sup\{x | F(x) < 1\}$, and $E(X)$ is finite for all x , where $1 < x < \infty$. Then, $E(X | X \geq x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and

$$h(x) = \frac{\Gamma(n)\Gamma(n-1)(x^2 + 1)^{2n}[E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x)]}{2\Gamma(2n)x^{2n-3}(x^2 - 1)},$$

if and only if

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, \quad x > 1, n \geq 2,$$

where $E(x)$ denotes the first moment given by Eq. (3.6), and $P(x)$ is given by Eq. (4.3).

Proof. Necessary Part: Consider the pdf (3.1), that is,

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}}, \quad x > 1, n \geq 2.$$

Then, since $E(X|X \geq x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$, and also since $E(X|X \geq x) = \frac{\int_x^\infty uf(u)du}{f(x)}$, we have

$$f(x)h(x) = \int_x^\infty uf(u)du = E(x) - \int_1^x uf(u)du = E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x), \quad (4.5)$$

where $E(x)$ denotes the first moment given by Eq. (3.6), and $P(x)$ is given by Eq. (4.3). It follows from (4.5) that

$$h(x) = \frac{\Gamma(n)\Gamma(n-1)(x^2+1)^{2n}[E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x)]}{2\Gamma(2n)x^{2n-3}(x^2-1)}. \quad (4.6)$$

This completes the proof of the necessary part. $\square$

Sufficient Part: Conversely, suppose that

$$h(x) = \frac{\Gamma(n)\Gamma(n-1)(x^2+1)^{2n}[E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x)]}{2\Gamma(2n)x^{2n-3}(x^2-1)}.$$

Then, differentiating both sides of the above equation with respect to x , and simplifying, we easily obtain

$$h'(x) = -x - h(x) \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right),$$

or,

$$\frac{x+h'(x)}{h(x)} = - \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right). \quad (4.7)$$

Since, by Lemma 4.2, we have

$$\frac{x+h'(x)}{h(x)} = -\frac{f'(x)}{f(x)},$$

therefore (4.7) reduces to

$$\frac{f'(x)}{f(x)} = \left(\frac{2n-3}{x} + \frac{4x}{x^2-1} - \frac{4nx}{x^2+1} \right). \quad (4.8)$$

On integrating both sides of the above Eq. (4.8) with respect to x , we obtain

$$\ln f(x) = \ln(c) + (2n-3)\ln(x) + 2\ln(x^2-1) - 2n\ln(x^2+1),$$

or,

$$f(x) = \frac{cx^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}}, \quad (4.9)$$

where c is a constant to be determined. Now, integrating the Eq. (4.9) with respect to x from 1 to ∞ , and using the boundary condition $\int_1^\infty f(x)dx = 1$, we obtain $c = \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}$. Then, we have

$$f(x) = \frac{2\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}}, \quad x > 1, n \geq 2.$$

This completes the proof of Theorem 4.2.

4.2 Characterizations by Order Statistics

If $X_1, X_2, \dots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah, et al. (2013), chapter 5, or Arnold, et al. (2005), chapter 2, that $X_{j,n}|X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j - k)$ th order statistics from $(n - k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \leq v < x$, and $X_{i,n}|X_{k,n} = x$, $1 \leq i < k \leq n$, is distributed as i th order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where

$$f_W(w|x) = \frac{f(w)}{F(x)}, w < x. \text{ Let } S_{k-1} = \frac{1}{k-1}(X_{1,n} + X_{2,n} + \dots + X_{k-1,n}),$$

and

$$T_{k,n} = \frac{1}{n-k}(X_{k+1,n} + X_{k+2,n} + \dots + X_{n,n}).$$

Theorem 4.3: Suppose the random variable X satisfies the Assumption 4.1 with $\gamma = 1$ and $\delta = \infty$, then $E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and

$$g(x) = \frac{(x^2 + 1)^{2n}P(x)}{2x^{2n-3}(x^2 - 1)^2},$$

where $P(x)$ is given by Eq. (4.3), if and only if

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, x > 1, n \geq 2.$$

Proof. It is known from see Ahsanullah, et al. (2013) and David and Nagaraja (2003) that $E(S_{k-1}|X_{k,n} = x) = E(X|X \leq x)$. Hence, by Theorem 4.1, the result follows. $\square$

Theorem 4.4: Suppose the random variable X satisfies the Assumption 4.1 with $\gamma = 1$ and $\delta = \infty$, then $E(T_{k,n}|X_{k,n} = x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and

$$h(x) = \frac{\Gamma(n)\Gamma(n-1)(x^2 + 1)^{2n}[E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x)]}{2\Gamma(2n)x^{2n-3}(x^2 - 1)},$$

if and only if

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2 - 1)^2}{(x^2 + 1)^{2n}}, x > 1, n \geq 2,$$

where $E(x)$ denotes the first moment given by Eq. (3.6), and $P(x)$ is given by Eq. (4.3).

Proof. It is known from see Ahsanullah, et al. (2013) and David and Nagaraja (2009) that $E(T_{k,n}|X_{k,n} = x) = E(X|X \geq x)$. Hence the proof follows from Theorem 4.2. $\square$

4.3 Characterization by Upper Record Values

For details on record values, see Ahsanullah (1995). Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, \dots, X_n)$ for $n \geq 1$ and $Y_j > Y_{j-1}$, $j > 1$, then X_j is called an upper record value of $\{X_n, n \geq 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min\{j|j > U(n+1), X_j > X_{U(n-1)}, n > 1\}\}$ and $U(1) = 1$. Let the n th upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 4.5: Suppose the random variable X satisfies the Assumption 4.1 with $\gamma = 1$ and $\delta = \infty$, then $E(X(n+1)|X(n) = x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and

$$h(x) = \frac{\Gamma(n)\Gamma(n-1)(x^2 + 1)^{2n}[E(x) - \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)}P(x)]}{2\Gamma(2n)x^{2n-3}(x^2 - 1)},$$

if and only if

$$f(x) = 2 \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n-1)} \frac{x^{2n-3}(x^2-1)^2}{(x^2+1)^{2n}}, x > 1, n \geq 2,$$

where $E(x)$ denotes the first moment given by Eq. (3.6), and $P(x)$ is given by Eq. (4.3).

Proof. It is known from Ahsanullah, et al. (2013) and Nevzorov (2001) that $E(X(n+1)|X(n)=x) = E(X|X \geq x)$. Then, the result is evident from Theorem 4.2. $\square$

5 Percentiles

For any statistical applications, it is important to know the percentage points of a given distribution. For example, one may be interested in knowing the median (50%), 25%, or 75% quartiles, or 5%, 90%, 95%, or 99% confidence levels for other applications, to assess the statistical significance of an observation whose distribution is known.

The 100 p th percentile or the quantile of order p , for any $0 < p < 1$, of our distribution with the pdf $f_X(x)$ as in Eq. (3.1) is defined as a number x_p such that the area under $f_X(x)$ to the left of x_p is p . In other words, x_p is any solution of the equation $F(x_p) = \int_1^{x_p} f_X(u) du = p$, where $F(x_p)$ denotes the cdf given by Eq. (3.6). Thus, by numerically solving the equation $F(x_p) = \int_1^{x_p} f_X(u) du = p$, using a Maple program, percentage points x_p associated with the cdf, $F(x_p)$, of X are computed for different sets of values of the parameters, $n \geq 2$, which are provided in the Table 3 below.

Table 3: Percentage Points

Percentiles n	0.75 (75 %)	0.80 (80 %)	0.85 (85 %)	0.90 (90 %)	0.95 (95 %)	0.99 (99 %)
2	4.5686	5.1866	6.0768	7.5467	10.8156	24.4335
3	2.8311	3.0573	3.3589	3.8084	4.6613	7.2155
4	2.3201	2.4617	2.6452	2.9085	3.3803	4.6382
5	2.0659	2.1714	2.3058	2.4950	2.8238	3.6485
6	1.9103	1.9956	2.1034	2.2530	2.5082	3.1242
7	1.8040	1.8762	1.9671	2.0921	2.3025	2.7973
8	1.7254	1.7889	1.8680	1.9763	2.1567	2.5725
9	1.6649	1.7217	1.7923	1.8884	2.0471	2.4076
10	1.6165	1.6682	1.7322	1.8189	1.9613	2.2809
11	1.5768	1.6243	1.6831	1.7624	1.8920	2.1800
12	1.5434	1.5876	1.6421	1.7154	1.8347	2.0976

6 Concluding Remarks

The condition numbers of Gaussian matrices occur in many branches of pure and applied sciences, and engineering. Many researchers have studied the distributions of condition numbers of Gaussian matrices. In this paper, several new distributional properties of the distribution of the condition number of a complex Gaussian matrix are presented. Based on the distributional properties, we have established some new characterization results. For some values of the parameter, $n \geq 2$, the first moment, $E(X)$, and the Shannon entropy are numerically computed. Since, for any statistical applications, it is important to know the percentage points of a given distribution, we also have computed the percentiles of the distribution. It is hoped that the findings of this paper will be quite useful for researchers in various fields of pure and applied

sciences.

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