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Kuratowski monoids of n -topological spaces

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Abstract: Generalizing the famous 14-set closure-complement Theorem of Kuratowski from 1922, we prove that for a set X endowed with n pairwise comparable topologies $\tau_1 \subset \dots \subset \tau_n$, by repeated application of the operations of complement and closure in the topologies $\tau_1, \dots, \tau_n$ to a subset $A \subset X$ we can obtain at most $2K(n) = 2 \sum_{i,j=0}^n \binom{i+j}{i} \binom{i+j}{j}$ distinct sets.

Keywords: Kuratowski monoid; polytopological space; n -topological space

MSC: 54A10, 54E55, 54H10, 06F05

1 Introduction

This paper was motivated by the famous Kuratowski 14-set closure-complement Theorem [4], which says that the repeated application of the operations of closure and complement to a subset A of a topological space X yields at most 14 pairwise distinct sets¹. More precisely, this theorem says that for any topological space (X, τ) the operators of complement $c : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $c : A \mapsto X \setminus A$ and closure $\bar{\tau} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $\bar{\tau} : A \mapsto \bar{A}$, generate a submonoid $\langle c, \bar{\tau} \rangle$ of cardinality ≤ 14 in the monoid $\mathcal{P}(X)^{\mathcal{P}(X)}$ of all self-maps of the power-set $\mathcal{P}(X)$ of X . The structure of this monoid was studied in [2], [3], [5], [7].

In [6] Shallitt and Willard constructed two commuting closure operators $p, q : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ on the power-set $\mathcal{P}(X)$ of a countable set X such that the submonoid $\langle p, q, c \rangle \subset \mathcal{P}(X)^{\mathcal{P}(X)}$ generated by these closure operators and the operator of complement is infinite. In Example 3.1 below we shall define two metrizable topologies τ_1 and τ_2 on a countable set X such that the closure operators $\bar{\tau}_1$ and $\bar{\tau}_2$ in the topologies τ_1 and τ_2 generate an infinite submonoid $\langle \bar{\tau}_1, \bar{\tau}_2 \rangle$ in the monoid $\mathcal{P}(X)^{\mathcal{P}(X)}$ of all self-maps of $\mathcal{P}(X)$. Moreover, for some set $A \subset X$ the set $\{f(A) : f \in \langle \bar{\tau}_1, \bar{\tau}_2 \rangle\}$ is infinite. This shows that Kuratowski's 14-set theorem does not generalize to spaces endowed with two or more topologies.

The situation changes dramatically if two topologies τ_1 and τ_2 on a set X are comparable, i.e., one of these topologies is contained in the other. In this case we shall prove that the closure operators $\bar{\tau}_1, \bar{\tau}_2 : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ induced by these topologies together with the operator c of complement generate a submonoid $\langle \bar{\tau}_1, \bar{\tau}_2, c \rangle \subset \mathcal{P}(X)^{\mathcal{P}(X)}$ of cardinality ≤ 126 . In fact, we shall consider this problem in a more general context of multitopological spaces and polytopological spaces.

By a *multitopological space* we understand a set X endowed with a family $\mathcal{T}$ of topologies on X . A multitopological space $(X, \mathcal{T})$ is called *polytopological* if the family of its topologies $\mathcal{T}$ is linearly ordered by the inclusion relation. A typical example of a polytopological space is the real line endowed with the Euclidean

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¹ A complete bibliography related to the Kuratowski 14-set closure-complement Theorem is collected on the web-site (<http://www.mathtransit.com/cornucopia.php>) created by Mark Bowron.

and Sorgenfrey topologies. Another natural example of a polytopological space is any Banach space, carrying the norm and weak topologies. A dual Banach space is an example of a polytopological space carrying three topologies: the norm topology, the weak topology and the $\ast$ -weak topology. A topological space (X, τ) can be thought as a polytopological space $(X, \{\tau\})$ endowed with the family $\{\tau\}$ consisting of a single topology τ .

For a topology τ on a set X by $\check{\tau} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and $\bar{\tau} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ we shall denote the operators of taking the interior and closure with respect to the topology τ . These operators assign to each subset $A \subset X$ its interior $\check{\tau}(A)$ and closure $\bar{\tau}(A)$, respectively. Since $\tau = \{\check{\tau}(A) : A \subset X\} = \{X \setminus \bar{\tau}(A) : A \subset X\}$ the topology τ can be recovered from the operators $\check{\tau}$ and $\bar{\tau}$.

For a multitopological space $\mathbf{X} = (X, \mathcal{T})$ the submonoid

$$K(\mathbf{X}) = \langle \check{\tau}, \bar{\tau} : \tau \in \mathcal{T} \rangle$$

in $\mathcal{P}(X)^{\mathcal{P}(X)}$ generated by the interior and closure operators $\check{\tau}, \bar{\tau}$ for $\tau \in \mathcal{T}$, will be called *the Kuratowski monoid* of the multitopological space $\mathbf{X}$. A somewhat larger submonoid

$$K_2(\mathbf{X}) = \langle c, \bar{\tau} : \tau \in \mathcal{T} \rangle$$

in $\mathcal{P}(X)^{\mathcal{P}(X)}$ generated by the operator of complement c and the closure operators $\bar{\tau}, \tau \in \mathcal{T}$, will be called *the full Kuratowski monoid* of the multitopological space $\mathbf{X}$.

Taking into account that $\check{\tau} = c \circ \bar{\tau} \circ c$ and $\bar{\tau} = c \circ \check{\tau} \circ c$, we see that $K(\mathbf{X}) \subset K_2(\mathbf{X})$ and moreover,

$$K_2(\mathbf{X}) = K(\mathbf{X}) \cup (c \circ K(\mathbf{X})),$$

which implies that $|K_2(\mathbf{X})| \leq 2 \cdot |K(\mathbf{X})|$.

The notion of a multitopological space has one disadvantage: multitopological spaces do not form a category (it is not clear what to understand under a morphism of multitopological spaces). This problem with multitopological spaces can be easily fixed by introducing their parametric version called L -topological spaces where $(L, \leq)$ is a partially ordered set.

Given a subset X we denote by $\text{Top}(X)$ the family of all possible topologies on X , partially ordered by the inclusion relation. The family $\text{Top}(X)$ is a lattice whose smallest element is the anti-discrete topology τ_a and the largest element is the discrete topology τ_d on X . Observe that for the discrete topology the operators $\check{\tau}_d$ and $\bar{\tau}_d$ coincide with the identity operator 1_X on $\mathcal{P}(X)$.

Let $(L, \leq)$ be a partially ordered set. By definition, an L -topology on a set X is any monotone map $\tau : L \rightarrow \text{Top}(X)$. The monotonicity of τ means that for any elements $i \leq j$ in L we get $\tau(i) \subset \tau(j)$. In the sequel for an element $i \in L$ it will be convenient to denote the topology $\tau(i)$ by τ_i . By an L -topological space we shall understand a pair (X, τ) consisting of a set X and an L -topology $\tau : L \rightarrow \text{Top}(X)$ on X .

By a morphism between two L -topological spaces (X, τ) and (Y, σ) we understand a map $f : X \rightarrow Y$ which is continuous as a map between topological spaces (X, τ_i) and (Y, σ_i) for every $i \in L$. L -Topological spaces and their morphisms form a category called the *category of L -topological spaces*. Each L -topological space $\mathbf{X} = (X, \tau)$ can be thought as a multitopological space endowed with the family of topologies $\{\tau_i\}_{i \in L}$. If the set L is linearly ordered, then the multitopological space $(X, \{\tau_i\}_{i \in L})$ is polytopological.

So we can speak about the Kuratowski monoid $K(\mathbf{X})$ and the full Kuratowski monoid $K_2(\mathbf{X})$ of an L -topological space $\mathbf{X}$.

We shall be especially interested in (full) Kuratowski monoids of n -topological spaces where $n = \{0, \dots, n-1\}$ is a finite non-zero ordinal (or a natural number). Observe that n -topological spaces can be thought as sets endowed with n -topologies $\tau_0 \subset \tau_1 \subset \dots \subset \tau_{n-1}$.

We shall prove that the upper bound for the cardinality of the Kuratowski monoid $K(X)$ of an n -topological space X is given by the number

$$K(n) = \sum_{i,j=0}^n \binom{i+j}{i} \cdot \binom{i+j}{j}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the binomial coefficient.

The main result of this paper is the following theorem.

Theorem 1.1. For any n -topological space $\mathbf{X} = (X, \mathcal{T})$ its Kuratowski monoid $K(\mathbf{X})$ has cardinality $|K(\mathbf{X})| \leq K(n)$ and its full Kuratowski monoid $K_2(\mathbf{X})$ has cardinality $|K_2(\mathbf{X})| \leq 2 \cdot K(n)$.

The upper bounds $|K(\mathbf{X})| \leq K(n)$ and $|K_2(\mathbf{X})| \leq 2 \cdot K(n)$ given in Theorem 1.1 are exact as shown in our next theorem.

Theorem 1.2. For every natural number n there exists an n -topological space $\mathbf{X} = (X, \mathcal{T})$ such that $|K(\mathbf{X})| = K(n)$ and $|K_2(\mathbf{X})| = 2 \cdot K(n)$.

This theorem will be proved in Section 8 (see Corollary 8.2). The asymptotics of the sequence $K(n)$ is described in the following theorem, which will be proved in Section 6.

Theorem 1.3. There exists $\lim_{n \rightarrow \infty} K(n) / \binom{2n}{n}^2 = \sup_{n \rightarrow \infty} K(n) / \binom{2n}{n}^2 = \frac{16}{9}$ which implies that $K(n) = \left(\frac{16}{9} + o(1)\right) \cdot \binom{2n}{n}^2 = \frac{16^{n+1}}{9\pi n} \cdot (1 + o(1))$.

The values of the sequences $K(n)$ and $2K(n)$ for $n \leq 10$, calculated with help of computer are presented in the following table:

n	0	1	2	3	4	5	6	7	8	9	10
$K(n)$	1	7	63	697	8549	111033	1495677	20667463	291020283	4157865643	60094478179
$2K(n)$	2	14	126	1394	17098	222066	2991354	41334926	582040566	8315731286	120188956358

In particular, the Kuratowski monoid $K(\mathbf{X})$ of any topological space $\mathbf{X} = (X, \tau)$ consists of 7 elements (some of which can coincide [2]):

$$1, \check{\tau}, \bar{\tau}, \check{\tau}\bar{\tau}, \bar{\tau}\check{\tau}, \check{\tau}\bar{\tau}\check{\tau}, \bar{\tau}\check{\tau}\bar{\tau}.$$

For a 2-topological space $\mathbf{X} = (X, \tau)$ endowed with two topologies $\tau_0 \subset \tau_1$ the number of elements of the Kuratowski monoid $K(\mathbf{X})$ increases to 63 (see Proposition 9.2):

$$\begin{aligned} &1, \check{\tau}_0, \check{\tau}_1, \bar{\tau}_0, \bar{\tau}_1, \\ &\check{\tau}_0\bar{\tau}_0, \check{\tau}_0\bar{\tau}_1, \check{\tau}_1\bar{\tau}_0, \check{\tau}_1\bar{\tau}_1, \bar{\tau}_0\check{\tau}_0, \bar{\tau}_0\check{\tau}_1, \bar{\tau}_1\check{\tau}_0, \bar{\tau}_1\check{\tau}_1, \\ &\check{\tau}_0\bar{\tau}_0\check{\tau}_0, \check{\tau}_0\bar{\tau}_0\check{\tau}_1, \check{\tau}_0\bar{\tau}_1\check{\tau}_0, \check{\tau}_0\bar{\tau}_1\check{\tau}_1, \check{\tau}_1\bar{\tau}_0\check{\tau}_0, \check{\tau}_1\bar{\tau}_0\check{\tau}_1, \check{\tau}_1\bar{\tau}_1\check{\tau}_0, \check{\tau}_1\bar{\tau}_1\check{\tau}_1, \\ &\bar{\tau}_0\check{\tau}_0\bar{\tau}_0, \bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \bar{\tau}_0\check{\tau}_1\bar{\tau}_0, \bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_0\bar{\tau}_0, \bar{\tau}_1\check{\tau}_0\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0, \bar{\tau}_1\check{\tau}_1\bar{\tau}_1, \\ &\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0, \check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \check{\tau}_1\bar{\tau}_0\check{\tau}_1\bar{\tau}_0, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_1, \\ &\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\bar{\tau}_1, \bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_0\bar{\tau}_0\bar{\tau}_0, \bar{\tau}_1\check{\tau}_0\bar{\tau}_0\bar{\tau}_1, \bar{\tau}_1\check{\tau}_0\bar{\tau}_1\bar{\tau}_0, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\bar{\tau}_0, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\bar{\tau}_1, \\ &\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\bar{\tau}_1, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\bar{\tau}_0, \check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\bar{\tau}_1, \\ &\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0, \bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \\ &\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1, \check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1, \\ &\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\bar{\tau}_1, \\ &\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\bar{\tau}_1, \bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1. \end{aligned}$$

2 The Kuratowski monoid of a saturated polytopological space

In this section we introduce a class of n -topological spaces $\mathbf{X} = (X, \mathcal{T})$ whose Kuratowski monoids $K(\mathbf{X})$ have cardinality strictly smaller than $K(n)$.

A multitopological space (X, τ) is called *saturated* if for any topologies $\tau_0, \tau_1 \in \mathcal{T}$ each non-empty open subset $U \in \tau_0$ has non-empty interior in the topology τ_1 . A typical example of a saturated multitopological space is the real line $\mathbb{R}$ endowed with the 2-element family $\mathcal{T} = \{\tau_0, \tau_1\}$ consisting of the Euclidean and Sorgenfrey topologies $\tau_0 \subset \tau_1$.

For a linearly ordered set L an L -topological space (X, τ) is defined to be *saturated* if the multitopological space $(X, \{\tau_i\}_{i \in L})$ is saturated.

Theorem 2.1. *For a saturated polytopological space $\mathbf{X} = (X, \mathcal{T})$ the Kuratowski monoid $K(\mathbf{X})$ coincides with the set*

$$\{1\} \cup \{\check{\tau}, \bar{\tau}, \check{\tau}\bar{\tau}, \bar{\tau}\check{\tau}, \check{\tau}\bar{\tau}\check{\tau}, \bar{\tau}\check{\tau}\bar{\tau} : \tau \in \mathcal{T}\}$$

and hence has cardinality $|K(\mathbf{X})| \leq 1 + 6 \cdot |\mathcal{T}|$.

Proof. We claim that $\bar{\tau}_0\check{\tau}_1 = \bar{\tau}_0\check{\tau}_0$ for any topologies $\tau_0, \tau_1 \in \mathcal{T}$. Separately we consider two cases.

First we assume that $\tau_0 \subset \tau_1$. Take any subset $A \subset X$ and observe that $\check{\tau}_0(A) \subset \check{\tau}_1(A)$ and hence $\bar{\tau}_0\check{\tau}_0(A) \subset \bar{\tau}_0\check{\tau}_1(A)$. On the other hand, for any point $x \in \bar{\tau}_0\check{\tau}_1(A)$ and any neighborhood $U_x \in \tau_0 \subset \tau_1$ of x the set $W = U_x \cap \check{\tau}_1(A) \in \tau_1$ is not empty and open in the topology τ_1 . Since the polytopological space $(X, \mathcal{T})$ is saturated, the set W contains some non-empty open set $V \in \tau_0$. Then $V \subset W \subset \check{\tau}_1(A) \subset A$ and hence $V \subset \check{\tau}_0(A)$ and $V \subset U_x \cap \check{\tau}_0(A)$, which implies that $x \in \bar{\tau}_0\check{\tau}_0(A)$ and finally $\bar{\tau}_0\check{\tau}_1(A) \subset \bar{\tau}_0\check{\tau}_0(A)$.

Next, assume that $\tau_1 \subset \tau_0$. Take any subset $A \subset X$ and observe that $\check{\tau}_1(A) \subset \check{\tau}_0(A)$ and hence $\bar{\tau}_0\check{\tau}_1(A) \subset \bar{\tau}_0\check{\tau}_0(A)$. On the other hand, for any point $x \in \bar{\tau}_0\check{\tau}_0(A)$ and any neighborhood $U_x \in \tau_0$ of x the set $W = U_x \cap \check{\tau}_0(A) \in \tau_0$ is not empty and open in the topology τ_0 . Since the polytopological space $(X, \mathcal{T})$ is saturated, the set W contains some non-empty open set $V \in \tau_1$. Then $V \subset W \subset \check{\tau}_0(A) \subset A$ and hence $V \subset \check{\tau}_1(A)$ and $V \subset U_x \cap \check{\tau}_1(A)$, which implies $x \in \bar{\tau}_0\check{\tau}_1(A)$ and finally $\bar{\tau}_0\check{\tau}_0(A) \subset \bar{\tau}_0\check{\tau}_1(A)$.

So, in both cases we have proven the equality $\bar{\tau}_0\check{\tau}_0 = \bar{\tau}_0\check{\tau}_1$. Applying to this equality the operator c of taking complement, we get

$$\check{\tau}_0\bar{\tau}_1 = c\bar{\tau}_0cc\check{\tau}_1c = c\bar{\tau}_0\check{\tau}_1c = c\bar{\tau}_0\check{\tau}_0c = c\bar{\tau}_0cc\check{\tau}_0c = \check{\tau}_0\bar{\tau}_0.$$

This implies that

$$K(\mathbf{X}) = \bigcup_{\tau \in \mathcal{T}} K(X, \tau) = \bigcup_{\tau \in \mathcal{T}} \{1, \check{\tau}, \bar{\tau}, \check{\tau}\bar{\tau}, \bar{\tau}\check{\tau}, \check{\tau}\bar{\tau}\check{\tau}, \bar{\tau}\check{\tau}\bar{\tau}\}$$

and hence $|K(\mathbf{X})| \leq 1 + \sum_{\tau \in \mathcal{T}} (|K(X, \tau)| - 1) \leq 1 + 6 \cdot |\mathcal{T}|$. $\square$

Example 2.2. Let $\mathbf{X} = (\mathbb{R}, \{\tau_0, \tau_1\})$ be the real line $\mathbb{R}$ endowed with the Euclidean topology τ_0 and the Sorgenfrey topology τ_1 . The Kuratowski monoid $K(\mathbf{X})$ has cardinality $|K(\mathbf{X})| = 13$. Moreover, for some set $A \subset \mathbb{R}$ the family $K(\mathbf{X})A = \{f(A) : A \in K(\mathbf{X})\}$ has cardinality $|K(\mathbf{X})A| = |K(\mathbf{X})| = 13$.

Proof. The upper bound $|K(\mathbf{X})| \leq 13$ follows from Theorem 2.1. To prove the lower bound $|K(\mathbf{X})| \geq 13$, consider the subset

$$A = \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}) \cup (1, 2) \cup \{3\} \cup ([4, 5) \cap \mathbb{Q})$$

and observe that the following 13 subsets of $\mathbb{R}$ are pairwise distinct, witnessing that $|K(\mathbf{X})| \geq |\{f(A) : f \in K(\mathbf{X})\}| \geq 13$.

A	$\bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}) \cup (1, 2) \cup \{3\} \cup ([4, 5) \cap \mathbb{Q})$
$\check{\tau}_0(A)$	$\bigcup_{n=0}^{\infty} (3^{-2n-1}, 3^{-2n}) \cup (1, 2)$
$\bar{\tau}_0(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2] \cup \{3\} \cup [4, 5]$
$\check{\tau}_0\check{\tau}_0(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2]$
$\check{\tau}_0\bar{\tau}_0(A)$	$\bigcup_{n=0}^{\infty} (3^{-2n-1}, 3^{-2n}) \cup [1, 2) \cup (4, 5)$
$\bar{\tau}_0\check{\tau}_0\bar{\tau}_0(A)$	$\bigcup_{n=0}^{\infty} (3^{-2n-1}, 3^{-2n}) \cup [1, 2)$
$\bar{\tau}_0\bar{\tau}_0\bar{\tau}_0(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2] \cup [4, 5]$
$\check{\tau}_1(A)$	$\bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}) \cup (1, 2)$
$\bar{\tau}_1(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2] \cup \{3\} \cup [4, 5]$
$\check{\tau}_1\check{\tau}_1(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2)$
$\check{\tau}_1\bar{\tau}_1(A)$	$\bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}) \cup [1, 2) \cup (4, 5)$
$\bar{\tau}_1\check{\tau}_1\bar{\tau}_1(A)$	$\bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}) \cup [1, 2)$
$\bar{\tau}_1\bar{\tau}_1\bar{\tau}_1(A)$	$\{0\} \cup \bigcup_{n=0}^{\infty} [3^{-2n-1}, 3^{-2n}] \cup [1, 2] \cup [4, 5]$

□

Theorem 2.1 gives a partial answer to the following general problem.

Problem 2.3. Which properties of a polytopological space X are reflected in the algebraic structure of its Kuratowski monoid $K(X)$?

Remark 2.4. For topological spaces this problem was studied in [1] and [2].

3 An example of a multitopological space with infinite Kuratowski monoid

In this section we shall construct the following example announced in the introduction.

Example 3.1. There is a countable space X endowed with two (incomparable) metrizable topologies τ_0, τ_1 such that the Kuratowski monoid $K(X)$ of the multitopological space $\mathbf{X} = (X, \{\tau_0, \tau_1\})$ is infinite. Moreover, for some set $A \subset X$ the family $\{f(A) : f \in K(X)\}$ is infinite.

Proof. Take any countable metrizable topological space X containing a decreasing sequence of non-empty subsets $(X_n)_{n \in \omega}$ such that $X_0 = X$, $\bigcap_{n \in \omega} X_n = \emptyset$ and X_{n+1} is nowhere dense in X_n for all $n \in \omega$.

To find such a space X , take the convergent sequence $S_0 = \{0\} \cup \{2^{-n} : n \in \omega\} \subset \mathbb{R}$, endowed with the subspace topology inherited from the real line, and consider the subspace

$$X = \{(x_k)_{k \in \omega} \in S_0^\omega : \exists n \in \omega \forall k \geq n \ x_k = 0\} \setminus \{0\}^\omega$$

of the countable power S_0^ω endowed with the Tychonoff product topology. It is easy to see that the subsets

$$X_n = \{(x_k)_{k \in \omega} \in X : \forall k < n \ x_k = 0\}, \quad n \in \omega,$$

have the required properties: $X_0 = X$, $\bigcap_{n \in \omega} X_n = \emptyset$ and X_{n+1} is nowhere dense in X_n .

On the space X consider two topologies

$$\tau_0 = \{U \subset X : \forall n \in \omega \ U \cap (X_{2n} \setminus X_{2n+2}) \text{ is open in } X_{2n} \setminus X_{2n+2}\}$$

and

$$\tau_1 = \{U \subset X : \forall n \in \omega \ U \cap (X_{2n+1} \setminus X_{2n+3}) \text{ is open in } X_{2n+1} \setminus X_{2n+3}\}.$$

Observe that τ_0 coincides with the topology of the topological sum $\bigoplus_{n \in \omega} X_{2n} \setminus X_{2n+2}$ while τ_1 coincides with the topology of the topological sum

$$\left(\bigoplus_{x \in X_0 \setminus X_1} \{x\} \right) \oplus \bigoplus_{n \in \omega} X_{2n+1} \setminus X_{2n+3}.$$

We claim that the multitopological space $\mathbf{X} = (X, \{\tau_0, \tau_1\})$ has the required properties.

By $\bar{\tau}_0, \bar{\tau}_1 : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ we denote the closure operators in the topologies τ_0 and τ_1 , respectively. The nowhere density of X_{n+1} in X_n for all $n \in \omega$ and the definition of the topologies τ_0, τ_1 imply that for every $n \in \omega$

- 1) $\bar{\tau}_0(X \setminus X_{2n+1}) = X \setminus X_{2n+2}$;
- 2) $\bar{\tau}_1(X \setminus X_{2n+2}) = X \setminus X_{2n+3}$;
- 3) $\bar{\tau}_1 \bar{\tau}_0(X \setminus X_{2n+1}) = X \setminus X_{2n+3}$;
- 4) $(\bar{\tau}_1 \bar{\tau}_0)^n(X \setminus X_1) = X \setminus X_{2n+1}$.

Therefore, for the set $A = X \setminus X_1$ the sets $(\bar{\tau}_1 \bar{\tau}_0)^n(A)$, $n \in \omega$, are pairwise distinct, which implies that the family $\{f(A) : f \in K(\mathbf{X})\}$ is infinite and hence the Kuratowski monoid $K(\mathbf{X})$ of the multitopological space $\mathbf{X} = (X, \{\tau_0, \tau_1\})$ is infinite too. $\square$

4 Kuratowski monoids

To prove Theorem 1.1 we shall use the natural structure of partial order on the monoid $\mathcal{P}(X)^{\mathcal{P}(X)}$. For two maps $f, g \in \mathcal{P}(X)^{\mathcal{P}(X)}$ we write $f \leq g$ if $f(A) \subset g(A)$ for every subset $A \subset X$. This partial order turns $\mathcal{P}(X)$ into a partially ordered monoid.

By a *partially ordered monoid* we understand a monoid M endowed with a partial order $\leq$ which is *compatible* with the semigroup operation of M in the sense that for any points $x, y, z \in M$ the inequality $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. Recall that a *monoid* is a semigroup S possessing a two-sided unit $1 \in S$.

Observe that for two comparable topologies $\tau_1 \subset \tau_2$ on a set X we get

$$\check{\tau}_1 \leq \check{\tau}_2 \leq 1_X \leq \bar{\tau}_2 \leq \bar{\tau}_1$$

where $1_X : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is the identity transformation of $\mathcal{P}(X)$. Now we see that for a polytopological space $\mathbf{X} = (X, \mathcal{T})$ its Kuratowski monoid $K(\mathbf{X}) = \langle \check{\tau}, \bar{\tau} : \tau \in \mathcal{T} \rangle$ is generated by the linearly ordered set

$$L(\mathbf{X}) = \{\check{\tau} : \tau \in \mathcal{T}\} \cup \{1_X\} \cup \{\bar{\tau} : \tau \in \mathcal{T}\}.$$

This leads to the following

Definition 4.1. A *Kuratowski monoid* is a partially ordered monoid K generated by a finite linearly ordered set L containing the unit 1 of K and consisting of idempotents.

The set L will be called a *linear generating set* of the Kuratowski monoid K . This set can be written as the union $L = L_- \cup \{1\} \cup L_+$ where $L_- = \{x \in L : x < 1\}$ and $L_+ = \{x \in L : x > 1\}$ are the sets of *negative* and *positive generating elements* of K .

A Kuratowski monoid K is called a *Kuratowski monoid of type (n, p)* if K has a linear generating set L such that $|L_-| = n$ and $|L_+| = p$.

For two numbers $n, p \in \omega$ consider the number

$$K(n, p) = \sum_{i=0}^n \sum_{j=0}^p \binom{i+j}{i} \cdot \binom{i+j}{j}$$

and observe that $K(n) = K(n, n)$ for every $n \in \omega$.

It is easy to see that for each polytopological space $\mathbf{X} = (X, \mathcal{T})$ endowed with $n = |\mathcal{T}|$ topologies, its Kuratowski monoid $K(\mathbf{X})$ is a Kuratowski monoid of type (n, n) or $(n-1, n-1)$. The latter case happens if the polytopology $\mathcal{T}$ of $\mathbf{X}$ contains the discrete topology τ_d on X . In this case $\check{\tau}_d = 1_X = \bar{\tau}_d$.

Now we see that Theorem 1.1 is a partial case of the following more general theorem, which will be proved in Section 7 (more precisely, in Theorem 7.1).

Theorem 4.2. *Each Kuratowski monoid K of type (n, p) has cardinality $|K| \leq K(n, p)$.*

The values of the double sequence $K(n, p)$ for $n, p \leq 9$ were calculated by computer:

$n \backslash p$	0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9	10
1	2	7	17	34	60	97	147	212	294	395
2	3	17	63	180	431	909	1743	3104	5211	8337
3	4	34	180	697	2173	5787	13677	29438	58770	110296
4	5	60	431	2173	8549	28039	80029	204690	479047	1041798
5	6	97	909	5787	28039	111033	376467	1128392	3059118	7629873
6	7	147	1743	13677	80029	376467	1495677	5192258	16140993	45761773
7	8	212	3104	29438	204690	1128392	5192258	20667463	73025423	233519803
8	9	294	5211	58770	479047	3059118	16140993	73025423	291020283	1042490763
9	10	395	8337	110296	1041798	7629873	45761773	233519803	1042490763	4157865643

5 Kuratowski words

Theorem 4.2 will be proved by showing that each element of a Kuratowski monoid K with linear generating set L can be represented by a Kuratowski word in the alphabet L . Kuratowski words are defined as follows.

By a *pointed linearly ordered set* we understand a linearly ordered set L with a distinguished element $1 \in L$ called the *unit* of L . This element divides the set $L \setminus \{1\}$ into negative and positive parts $L_- = \{x \in L : x < 1\}$ and $L_+ = \{x \in L : x > 1\}$, respectively. By $FS_L = \bigcup_{n=1}^{\infty} L^n$ we denote the free semigroup over L . It consists of non-empty words in the alphabet L . The semigroup operation on FS_L is defined as the concatenation of words. The set L is identified with the set L^1 of words of length 1 in the alphabet L .

A word $w = x_1 \dots x_n \in FS_L$ of length n is called *alternating* if for each natural number i with $1 \leq i < n$ the doubleton $\{x_i, x_{i+1}\}$ intersects both sets L_- and L_+ . According to this definition, words of length 1 also are alternating. On the other hand, an alternating word of length ≥ 2 does not contain a letter equal to 1.

An alternating word $x_0 \dots x_n \in FS_L$ of length $n+1 \geq 2$ is defined to be

- a $V_{\mp}$ -word iff there is an integer number $m \in \{0, \dots, n-1\}$ such that the sequences $(x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}}$, $(x_{m-2i})_{0 \leq i \leq \frac{m}{2}}$ are strictly increasing in L_- and the sequences $(x_{m+1+2i})_{0 \leq i < \frac{n-m}{2}}$, $(x_{m+1-2i})_{0 \leq i \leq \frac{m+1}{2}}$ are strictly decreasing in L_+ ;
- a $V_{\pm}$ -word if there is a number m such that $m \in \{0, \dots, n-1\}$ such that the sequences $(x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}}$, $(x_{m-2i})_{0 \leq i \leq \frac{m}{2}}$ are strictly decreasing in L_+ and the sequences $(x_{m+1+2i})_{0 \leq i < \frac{n-m}{2}}$, $(x_{m+1-2i})_{0 \leq i \leq \frac{m+1}{2}}$ are strictly increasing in L_- ;
- a W_- -word if there is a number $m \in \{1, \dots, n-1\}$ such that $x_{m-1} = x_{m+1} \in L_-$, the sequences $(x_{m+1+2i})_{0 \leq i < \frac{n-m}{2}}$, $(x_{m-1-2i})_{0 \leq i \leq \frac{m-1}{2}}$ are strictly increasing in L_- and the sequences $(x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}}$, $(x_{m-2i})_{0 \leq i \leq \frac{m}{2}}$ are strictly decreasing in L_+ ;
- a W_+ -word if there is a number $m \in \{1, \dots, n-1\}$ such that $x_{m-1} = x_{m+1} \in L_+$, the sequences $(x_{m+1+2i})_{0 \leq i < \frac{n-m}{2}}$, $(x_{m-1-2i})_{0 \leq i \leq \frac{m-1}{2}}$ are strictly decreasing in L_+ and the sequences $(x_{m+2i})_{0 \leq i \leq \frac{n-m}{2}}$, $(x_{m-2i})_{0 \leq i \leq \frac{m}{2}}$ are strictly increasing in L_- .

By $\mathcal{V}_{\mp}$ (resp. $\mathcal{V}_{\pm}$, $\mathcal{W}_{-}$, $\mathcal{W}_{+}$) we denote the family of all $V_{\mp}$ -words (resp. $V_{\pm}$ -words, W_{-} -word, W_{+} -words) in the alphabet L . It is easy to see that the families of words $\mathcal{V}_{\mp}$, $\mathcal{V}_{\pm}$, $\mathcal{W}_{-}$, $\mathcal{W}_{+}$ are pairwise disjoint. Words that belong to the set

$$\mathcal{K}_L = L^1 \cup \mathcal{V}_{\pm} \cup \mathcal{V}_{\mp} \cup \mathcal{W}_{-} \cup \mathcal{W}_{+}$$

are called *Kuratowski words* in the alphabet L .

Let us calculate the cardinality of the set $\mathcal{K}_L$ depending on the cardinalities $n = |L_{-}|$ and $p = |L_{+}|$ of the negative and positive parts of L .

For non-negative integers n, r by $\binom{n}{r}$ we denote the cardinality of the set of r -element subsets of an n -element set. It is clear that

$$\binom{n}{r} = \begin{cases} \frac{n!}{r!(n-r)!} & \text{if } 0 \leq r \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

The numbers $\binom{n}{r}$ will be called binomial coefficients. The following properties of binomial coefficients are well-known (see, e.g. [3, §5.1]).

Lemma 5.1. *For any non-negative integer numbers m, n, k we get*

- 1) $\binom{n}{k} = \binom{n}{n-k}$,
- 2) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$, and
- 3) $\binom{n+m}{k} = \sum_{l=0}^n \binom{n}{l} \binom{m}{k-l}$.

In the following theorem we calculate the cardinality of the set $\mathcal{K}_L$ of Kuratowski words.

Theorem 5.2. *For any finite pointed linearly ordered set L with $n = |L_{-}|$ and $p = |L_{+}|$ we get*

- 1) $|\mathcal{V}_{\mp}| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l+1} \binom{b+1}{r}$,
- 2) $|\mathcal{V}_{\pm}| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l} \binom{b+1}{r+1}$,
- 3) $|\mathcal{W}_{+}| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l} \binom{b+1}{r}$,
- 4) $|\mathcal{W}_{-}| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l+1} \binom{b+1}{r+1}$,
- 5) $|\mathcal{K}_L| = \sum_{a=0}^n \sum_{b=0}^p \binom{a+b}{a} \binom{a+b}{b} = K(n, p)$.

Proof. 1. To calculate the number of $V_{\mp}$ -words, fix any $V_{\mp}$ -word $v \in V_{\mp}$ and write it as an alternating word $v = x_k \dots x_{2m} \dots x_q$ such that the sequences $(x_{2m+2i})_{0 \leq i \leq \frac{q-2m}{2}}$ and $(x_{2m-2i})_{0 \leq i \leq \frac{2m-k}{2}}$ are strictly increasing in L_{-} and the sequences $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ and $(x_{2m+1-2i})_{0 \leq i \leq \frac{2m+1-k}{2}}$ are strictly decreasing in L_{+} . It follows that the sequence $(x_{2m+2i})_{1 \leq i \leq \frac{q-2m}{2}}$ is a strictly increasing sequence of length $r = \lfloor \frac{q-2m}{2} \rfloor$ in the linearly ordered set $A = \{x \in L : x > x_{2m}\} \subset L_{-}$. The number of such sequences is equal to $\binom{a}{r}$ where the cardinality $a = |A|$ can vary from 0 (if x_{2m} is the largest element of the set L_{-}) till $n-1$ (if x_{2m} is the smallest element of L_{-}). By analogy, $(x_{2m-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly increasing sequence of length $l = \lfloor \frac{2m-k}{2} \rfloor$ in the linearly ordered set A and the number of such sequences is equal to $\binom{a}{l}$.

If $l = \frac{2m-k}{2}$, then $2m = k+2l$ and $(x_{2m+1-2i})_{1 \leq i \leq \frac{2m+1-k}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{2m+1-k}{2} \rfloor = l$ in the linearly ordered set $B = \{x \in L : x < x_{2m+1}\} \subset L_{+}$. The number of such sequences is equal to $\binom{b}{l}$ where $b = |B| < p$. If $l = \frac{2m-k-1}{2}$, then $2m = k+1+2l$ and $(x_{2m+1-2i})_{1 \leq i \leq \frac{2m+1-k}{2}}$ is a strictly increasing sequence of length $\lfloor \frac{2m+1-k}{2} \rfloor = l+1$ in the set B . The number of such sequences is equal to $\binom{b}{l+1}$.

If $r = \frac{q-2m}{2}$, then $2m = q-2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{q-2m-1}{2} \rfloor = r-1$ in the linearly ordered set B . The number of such sequences is equal to $\binom{b}{r-1}$. If $r = \frac{q-2m-1}{2}$, then $2m = q-1-2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequences of length $\lfloor \frac{q-2m-1}{2} \rfloor = r$ in the linearly

ordered set B . The number of such sequences is equal to $\binom{b}{r}$. Summing up and applying Lemma 5.1(2), we conclude that the family $\mathcal{V}_\mp$ of all $V_\mp$ -words has cardinality

$$\begin{aligned} |\mathcal{V}_\mp| &= \sum_{a=0}^{n-1} \sum_{l=0}^a \sum_{r=0}^a \sum_{b=0}^{p-1} \binom{a}{l} \binom{a}{r} \left(\binom{b}{l} \binom{b}{r} + \binom{b}{l+1} \binom{b}{r} + \binom{b}{l} \binom{b}{r-1} + \binom{b}{l+1} \binom{b}{r-1} \right) = \\ &= \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \left(\binom{b+1}{l+1} \binom{b}{r} + \binom{b+1}{l+1} \binom{b}{r-1} \right) = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l+1} \binom{b+1}{r}. \end{aligned}$$

2. By analogy we can prove that

$$|\mathcal{V}_\pm| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l} \binom{b+1}{r+1}.$$

3. To calculate the number of W_+ -words, fix any W_+ -word $w \in \mathcal{W}_+$ and write it as an alternating word $w = x_k \dots x_{2m} \dots x_q$ such that $k < 2m < q$, $x_{2m} \in L_-$, $x_{2m-1} = x_{2m+1} \in L_+$, the sequences $(x_{2m+2i})_{0 \leq i \leq \frac{q-2m}{2}}$ and $(x_{2m-2i})_{0 \leq i \leq \frac{2m-k}{2}}$ are strictly increasing in L_- whereas the sequences $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ and $(x_{2m-1-2i})_{0 \leq i \leq \frac{2m-k-1}{2}}$ are strictly decreasing in L_+ .

It follows that $(x_{2m-2i})_{1 \leq i \leq \frac{2m-k}{2}}$ is a strictly increasing sequence of length $l = \lfloor \frac{2m-k}{2} \rfloor$ in the linearly ordered set $A = \{x \in L : x > x_{2m}\} \subset L_-$. The number of such sequences is equal to $\binom{a}{l}$ where $a = |A| < n$. By analogy, $(x_{2m+2i})_{0 \leq i \leq \frac{q-2m}{2}}$ is a strictly increasing sequence of length $r = \lfloor \frac{q-2m}{2} \rfloor$ in the linearly ordered set A and the number of such sequences is equal to $\binom{a}{r}$.

If $l = \frac{2m-k}{2}$, then $2m = 2l + k$ and $(x_{2m-1-2i})_{1 \leq i \leq \frac{2m-k-1}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{2m-k-1}{2} \rfloor = \lfloor \frac{2l-1}{2} \rfloor = l-1$ in the linearly ordered set $B = \{x \in L : x < x_{2m-1} = x_{2m+1}\} \subset L_+$. The number of such sequences is equal to $\binom{b}{l-1}$ where $b = |B| < p$. If $l = \frac{2m-k-1}{2}$, then $(x_{2m-1-2i})_{1 \leq i \leq \frac{2m-k-1}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{2m-k-1}{2} \rfloor = l$ in the set B . The number of such sequences is equal to $\binom{b}{l}$.

If $r = \frac{q-2m}{2}$, then $2m = q - 2r$ and $(x_{2m+1+2i})_{0 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{q-2m-1}{2} \rfloor = r-1$ in the linearly ordered set B . The number of such sequences is equal to $\binom{b}{r-1}$. If $r = \frac{q-2m-1}{2}$, then $2m = q - 1 - 2r$ and $(x_{2m+1+2i})_{1 \leq i \leq \frac{q-2m-1}{2}}$ is a strictly decreasing sequence of length $\lfloor \frac{q-2m-1}{2} \rfloor = r$ in the linearly ordered set B . The number of such sequences is equal to $\binom{b}{r}$. Summing up, we conclude that the family $\mathcal{W}_+$ of all W_+ -words has cardinality

$$\begin{aligned} |\mathcal{W}_+| &= \sum_{a=0}^{n-1} \sum_{l=0}^a \sum_{r=0}^a \sum_{b=0}^{p-1} \binom{a}{l} \binom{a}{r} \left(\binom{b}{l} \binom{b}{r} + \binom{b}{l} \binom{b}{r-1} + \binom{b}{l-1} \binom{b}{r} + \binom{b}{l-1} \binom{b}{r-1} \right) = \\ &= \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \left(\binom{b}{l} \binom{b+1}{r} + \binom{b}{l-1} \binom{b+1}{r} \right) = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l} \binom{b+1}{r}. \end{aligned}$$

4. The equality $|\mathcal{W}_-| = \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+1}{l+1} \binom{b+1}{r+1}$ can be proved by analogy with the preceding proof.

5. By the preceding items

$$\begin{aligned}
 |\mathcal{K}_L| &= |L| + |\mathcal{V}_\mp| + |\mathcal{V}_\pm| + |\mathcal{W}_-| + |\mathcal{W}_+| = 1 + n + p + |\mathcal{V}_\mp| + |\mathcal{W}_-| + |\mathcal{V}_\pm| + |\mathcal{W}_+| = \\
 &= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \left(\binom{b+1}{l+1} \binom{b+1}{r} + \binom{b+1}{l} \binom{b+1}{r+1} + \binom{b+1}{l} \binom{b+1}{r+1} + \binom{b+1}{l+1} \binom{b+1}{r} \right) = \\
 &= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \left(\binom{b+2}{l+1} \binom{b+1}{r} + \binom{b+2}{l+1} \binom{b+1}{r+1} \right) = \\
 &= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \sum_{l=0}^a \sum_{r=0}^a \binom{a}{l} \binom{a}{r} \binom{b+2}{l+1} \binom{b+2}{r+1} = \\
 &= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \left(\sum_{l=0}^a \binom{a}{l} \binom{b+2}{l+1} \right)^2 = 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \left(\sum_{l=0}^a \binom{a}{l} \binom{b+2}{b+1-l} \right)^2 = \\
 &= 1 + n + p + \sum_{a=0}^{n-1} \sum_{b=0}^{p-1} \binom{a+b+2}{b+1}^2 = 1 + n + p + \sum_{a=1}^n \sum_{b=1}^p \binom{a+b}{b}^2 = \sum_{a=0}^n \sum_{b=0}^p \binom{a+b}{b} \binom{a+b}{a} = K(n, p).
 \end{aligned}$$

□

6 An asymptotics of the sequence $K(n)$

In this section we study the asymptotical growth of the sequence $K(n) = K(n, n)$ and prove Theorem 1.3 announced in the Introduction as a corollary of the following results.

For every integers $0 \leq a, b \leq n$ put

$$c_{a,b}(n) = \frac{\binom{2n-a-b}{n-a}}{\binom{2n}{n}} = \frac{n(n-1) \cdots (n-a+1) \cdot n(n-1) \cdots (n-b+1)}{2n(2n-1) \cdots (2n-a-b+1)}$$

and observe that

$$\lim_{n \rightarrow \infty} c_{a,b}(n) = 2^{-(a+b)}. \quad (1)$$

For every $n \geq 0$ put $k(n) = K(n) / \binom{2n}{n}^2$ and observe that

$$k(n) = \frac{K(n)}{\binom{2n}{n}^2} = \sum_{i,j=0}^n \frac{\binom{i+j}{i}^2}{\binom{2n}{n}^2} = \sum_{a,b=0}^n \frac{\binom{n-a+n-b}{n-a}^2}{\binom{2n}{n}^2} = \sum_{a,b=0}^n c_{a,b}(n)^2.$$

Proposition 6.1. For every $n \geq 0$ we have $k(n) \leq \frac{16}{9}$.

Proof. $k(0) = 1$, $k(1) = k(2) = \frac{7}{4} < \frac{16}{9}$, $k(3) = \frac{697}{400} < \frac{7}{4}$, and $k(4) = \frac{8549}{4900} < \frac{16}{9}$. Suppose now that for some $n \geq 5$ we have proved that $k(n-1) \leq \frac{16}{9}$.

We shall use the following two lemmas.

Lemma 6.2. For each $0 < a \leq n$ we have $c_{a,0}(n) < 2^{-a}$.

Proof. This lemma follows from the equality $c_{a,0}(n) = \frac{n(n-1) \cdots (n-a+1)}{2n(2n-1) \cdots (2n-a+1)}$ and the inequality $(2n-l) > 2(n-l)$ holding for all $0 < l \leq n$. □

Lemma 6.3. $\frac{16}{9} c_{1,1}(n)^2 + 2(c_{1,0}(n)^2 + c_{2,0}(n)^2) < \frac{1}{9} + 2\left(\frac{1}{4} + \frac{1}{4^2}\right)$ for $n \geq 5$.

Proof. Observe that $c_{1,1}(n) = \frac{n}{2(2n-1)}$, $c_{1,0}(n) = 1/2$, and $c_{2,0}(n) = \frac{n-1}{2(2n-1)}$. Routine transformations show that the inequality in the lemma are equivalent to $19 < 4n$, which holds for $n \geq 5$. □

Now we have that

$$\begin{aligned}
 k(n) &= c_{0,0}(n)^2 + 2 \sum_{a=1}^n c_{a,0}(n)^2 + \sum_{a,b=1}^n c_{a,b}(n)^2 = 1 + 2 \sum_{a=1}^n c_{a,0}(n)^2 + \sum_{a,b=0}^{n-1} c_{a+1,b+1}(n)^2 = \\
 &= 1 + 2 \sum_{a=1}^n c_{a,0}(n)^2 + \sum_{a,b=0}^{n-1} \frac{n^4}{(2n(2n-1))^2} c_{a,b}(n-1)^2 = 1 + 2 \sum_{a=3}^n c_{a,0}(n)^2 + 2 \sum_{a=1}^2 c_{a,0}(n)^2 + c_{1,1}(n)^2 k(n-1) < \\
 &< 1 + 2 \sum_{a=3}^n \frac{1}{4^a} + 2(c_{1,0}(n)^2 + c_{2,0}(n)^2) + c_{1,1}(n)^2 \cdot \frac{16}{9} < 1 + 2 \sum_{a=3}^{\infty} \frac{1}{4^a} + 2\left(\frac{1}{4} + \frac{1}{4^2}\right) + \frac{1}{9} = \frac{16}{9}
 \end{aligned}$$

according to Lemma 6.3. □

Proposition 6.4. *There exists a limit $\lim_{n \rightarrow \infty} k(n) = 16/9$.*

Proof. The equality (1) implies that

$$\lim_{n \rightarrow \infty} k(n) \geq \sum_{a,b=0}^{\infty} 2^{-2(a+b)} = \sum_{i=0}^{\infty} (i+1)4^{-i} = 16/9.$$

By Proposition 6.1, $\lim_{n \rightarrow \infty} k(n) = 16/9$. □

By Stirling's approximation, $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + o(1))$, which yields the second equality in Theorem 1.3.

7 Representing elements of Kuratowski monoids by Kuratowski words

Let K be a Kuratowski monoid with linear generating set L and let $L_- = \{x \in L : x < 1\}$ and $L_+ = \{x \in L : x > 1\}$ be the negative and positive parts of L , respectively. Let $FS_L = \bigcup_{n=1}^{\infty} L^n$ be the free semigroup over L and $\pi : FS_L \rightarrow K$ be the homomorphism assigning to each word $x_1 \dots x_n \in FS_L$ the product $x_1 \dots x_n$ of its letters in K . The homomorphism $\pi : FS_L \rightarrow K$ induces a congruence $\sim$ on FS_L which identifies two words $u, v \in FS_L$ iff $\pi(u) = \pi(v)$.

A word $w \in FS_L$ is called *irreducible* if w has the smallest possible length in its equivalence class $[w]_{\sim} = \{u \in FS_L : u \sim w\}$. Since the set of natural numbers is well-ordered, for each element $x \in K$ there is an irreducible word $w \in FS_L$ such that $x = \pi(w)$. Consequently, the cardinality of K does not exceed the cardinality of the set of irreducible words in FS_L .

Theorem 7.1. *Each irreducible word in FS_L is a Kuratowski word. Consequently, $\pi(\mathcal{K}_L) = K$ and $|K| \leq |\mathcal{K}_L|$. If the set L is finite, then $|K| \leq |\mathcal{K}_L| = K(n, p)$ where $n = |L_-|$ and $p = |L_+|$.*

Proof. We divide the proof of Theorem 7.1 into a series of lemmas.

Lemma 7.2. *For any elements $x, y \in L_- \cup \{1\}$ we get $xy = \min\{x, y\}$.*

Proof. Since L is linearly ordered, either $x \leq y$ or $y \leq x$.

If $x \leq y$, then multiplying this inequality by x , we get $x = xx \leq xy$. On the other hand, multiplying the inequality $y \leq 1$ by x , we get the reverse inequality $xy \leq x1 = x$. Taking into account that $x \leq xy \leq x$, we conclude that $x = xy$.

If $y \leq x$, then multiplying this inequality by y , we get $y = yy \leq xy$. On the other hand, multiplying the inequality $x \leq 1$ by y , we get $xy \leq 1y = y$. Taking into account that $y \leq xy \leq y$, we conclude that $xy = y = \min\{x, y\}$. □

By analogy we can prove:

Lemma 7.3. *For any elements $x, y \in L_+ \cup \{1\}$ we get $xy = \max\{x, y\}$.*

Proof. If $x \leq y$, then multiplying this inequality by y , we obtain $xy \leq yy = y$. On the other hand, multiplying the inequality $1 \leq x$ by y , we get $y = 1y \leq xy$. So, $xy = y = \max\{x, y\}$.

If $y \leq x$, then after multiplication by x , we obtain $xy \leq xx = x$. On the other hand, multiplying the inequality $1 \leq y$ by x , we get $x \leq xy$ and hence $xy = x = \max\{x, y\}$. $\square$

Recall that a word $x_1 \dots x_n \in FS_L$ is *alternating* if for each natural number i with $1 \leq i < n$ the doubleton $\{x_i, x_{i+1}\}$ intersects both sets L_- and L_+ . According to this definition, one-letter words also are alternating. Lemmas 7.2 and 7.3 imply:

Lemma 7.4. *Each irreducible word $w \in FS_L$ is alternating.*

The following lemma will help us to reduce certain alternating words of length 4.

Lemma 7.5. *If $x_1x_2x_3x_4 \in FS_L$ is an alternating word in the alphabet L such that $x_1x_3 = x_1$ and $x_2x_4 = x_4$ in K , then $x_1x_2x_3x_4 = x_1x_4$ in K and hence $x_1x_2x_3x_4 \sim x_1x_4$.*

Proof. Two cases are possible.

1) $x_1, x_3 \in L_-$ and $x_2, x_4 \in L_+$. In this case the equalities $x_1x_3 = x_1$ and $x_2x_4 = x_4$ imply that $x_1 \leq x_3$ and $x_2 \leq x_4$ (see Lemmas 7.2 and 7.3). To see that $x_1x_2x_3x_4 = x_1x_4$, observe that

$$x_1x_2x_3x_4 \leq x_1x_2 \cdot 1 \cdot x_4 = x_1x_2x_4 = x_1x_4.$$

On the other hand,

$$x_1x_4 = (x_1x_3)x_4 = x_1 \cdot 1 \cdot x_3x_4 \leq x_1x_2x_3x_4.$$

These two inequalities imply the desired equality $x_1x_2x_3x_4 = x_1x_4$.

2) $x_1, x_3 \in L_+$ and $x_2, x_4 \in L_-$. In this case the equalities $x_1x_3 = x_1$ and $x_2x_4 = x_4$ imply that $x_1 \geq x_3$ and $x_2 \geq x_4$ (see Lemmas 7.2 and 7.3). To see that $x_1x_2x_3x_4 = x_1x_4$, observe that

$$x_1x_2x_3x_4 \leq x_1 \cdot 1 \cdot x_3x_4 = x_1x_3x_4 = x_1x_4.$$

On the other hand,

$$x_1x_4 = x_1(x_2x_4) = x_1x_2 \cdot 1 \cdot x_4 \leq x_1x_2x_3x_4.$$

These two inequalities imply the desired equality $x_1x_2x_3x_4 = x_1x_4$. $\square$

Now we are able to prove that each irreducible word $w \in FS_L$ is a Kuratowski word. If w consists of a single letter, then it is trivially Kuratowski and we are done. So, we assume that w has length ≥ 2 . By Lemma 7.4 the word w is alternating and hence can be written as the product $w = x_k \dots x_n$ for some $k \in \{0, 1\}$ and $n > k$ such that $x_{2i} \in L_-$ for all integer numbers i with $k \leq 2i \leq n$, and $x_{2i-1} \in L_+$ for all integer numbers i with $k \leq 2i-1 \leq n$.

Let m be the smallest number such that $k \leq 2m \leq n$ and

$$x_{2m} = \min\{x_{2i} : k \leq 2i \leq n\}$$

in L_- . First we shall analyze the structure of the subword $x_k \dots x_{2m}$ of the word $w = x_k \dots x_n$.

Lemma 7.6. *The sequence $(x_{2m-2i})_{0 \leq i \leq \frac{2m-k}{2}}$ is strictly increasing in L_- .*

Proof. Assume conversely that $x_{2i} \geq x_{2i-2}$ for some number $i \leq m$ and assume that i is the largest possible number with this property. The definition of the number m guarantees that $i < m$. Consequently, $x_{2i-2} \leq x_{2i}$ and $x_{2i} > x_{2i+2}$. Taking into account that $x_{2i-2}, x_{2i}, x_{2i+2} \in L_-$ and applying Lemma 7.2, we get $x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2}$ and $x_{2i}x_{2i+2} = \min\{x_{2i}, x_{2i+2}\} = x_{2i+2}$.

Now consider the elements $x_{2i-1}, x_{2i+1} \in L_+$. If $x_{2i-1} \leq x_{2i+1}$, then $x_{2i-1}x_{2i+1} = x_{2i+1}$ by Lemma 7.3 and by Lemma 7.5 the alternating word $x_{2i-2}x_{2i-1}x_{2i}x_{2i+1} = x_{2i-2}x_{2i+1}$ is reducible as $x_{2i-2}x_{2i} = x_{2i-2}$ and $x_{2i-1}x_{2i+1} = x_{2i+1}$.

If $x_{2i-1} > x_{2i+1}$, then $x_{2i-1}x_{2i+1} = x_{2i-1}$ by Lemma 7.3. Lemma 7.5 guarantees that the alternating word $x_{2i-1}x_{2i}x_{2i+1}x_{2i+2} = x_{2i-1}x_{2i+2}$ is reducible as $x_{2i-1}x_{2i+1} = x_{2i-1}$ and $x_{2i}x_{2i+2} = x_{2i+2}$.

Therefore, the word $x_k \cdots x_n$ contains a reducible subword and hence is reducible, which contradicts the choice of this word. This contradiction shows that $x_{2i-2} < x_{2i}$ and hence $x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2} \neq x_{2i}$ according to Lemma 7.2. $\square$

Lemma 7.7. *The sequence $(x_{2m-1-2i})_{0 \leq i \leq \frac{2m-k-1}{2}}$ is strictly decreasing in L_+ .*

Proof. Assume conversely that $x_{2i-1} \geq x_{2i+1}$ for some number $i < m$ with $k \leq 2i - 1$. Since $x_{2i-1}, x_{2i+1} \in L_+$, Lemma 7.3 implies that $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1}$. By Lemma 7.6, $x_{2i}x_{2i+2} = x_{2i+2}$. Then by Lemma 7.5 the alternating word $x_{2i-1}x_{2i}x_{2i+1}x_{2i+2}$ is equal to $x_{2i-1}x_{2i+2}$. This implies that the word $x_k \cdots x_n$ is reducible, which contradicts the choice of this word. This contradiction shows that $x_{2i-1} < x_{2i+1}$ and hence $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1} \neq x_{2i-1}$ according to Lemma 7.3. $\square$

Next, we consider the subword $x_{2m+1} \cdots x_n$ of the word $w = x_k \cdots x_n$.

Lemma 7.8. *The sequence $(x_{2m+2i})_{1 \leq i \leq \frac{n-2m}{2}}$ is strictly increasing in L_- .*

Proof. Assume conversely that $x_{2i} \geq x_{2i+2}$ for some number $i > m$ with $2i + 2 \leq n$. We can assume that i is the smallest possible number with this property. Then either $i = m + 1$ or else $x_{2i-2} < x_{2i}$. If $i = m + 1$, then $x_{2m} = x_{2i-2} \leq x_{2i}$ by the choice of m . In both cases we get $x_{2i-2} \leq x_{2i}$, which implies $x_{2i-2}x_{2i} = \min\{x_{2i-2}, x_{2i}\} = x_{2i-2}$ according to Lemma 7.2. The same Lemma 7.2 implies that $x_{2i}x_{2i+2} = \min\{x_{2i}, x_{2i+2}\} = x_{2i+2}$.

Now consider the elements $x_{2i-1}, x_{2i+1} \in L_+$. If $x_{2i-1} \geq x_{2i+1}$, then $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1}$ and the alternating word $x_{2i-1}x_{2i}x_{2i+1}x_{2i+2} = x_{2i-1}x_{2i+2}$ is reducible according to Lemma 7.5. If $x_{2i-1} < x_{2i+1}$, then $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1}$ and the alternating word $x_{2i-2}x_{2i-1}x_{2i}x_{2i+1} = x_{2i-2}x_{2i+1}$ is reducible by Lemma 7.5. But this contradicts the irreducibility of the word $x_k \cdots x_n$. So, $x_{2i} < x_{2i+2}$ and $x_{2i}x_{2i+2} = \min\{x_{2i}, x_{2i+2}\} = x_{2i} \neq x_{2i+2}$ according to Lemma 7.2. $\square$

Lemma 7.9. *The sequence $(x_{2m+1+2i})_{0 \leq i \leq \frac{n-2m-1}{2}}$ is strictly decreasing in L_+ .*

Proof. Assume conversely that $x_{2i-1} \leq x_{2i+1}$ for some $i > m$ with $2i + 1 \leq n$. Then $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i+1}$ according to Lemma 7.3. If $i = m + 1$, then $x_{2i-2}x_{2i} = x_{2m}x_{2i} = \min\{x_{2m}, x_{2i}\} = x_{2m} = x_{2i-2}$ by the choice of the number m . If $i > m + 1$, then $x_{2i-2}x_{2i} = x_{2i-2}$ by Lemma 7.8. By Lemma 7.5 the alternating word $x_{2i-2}x_{2i-1}x_{2i}x_{2i+1} = x_{2i-2}x_{2i+1}$ is reducible, which contradicts the irreducibility of the word $x_k \cdots x_n$. This contradiction shows that $x_{2i-1} > x_{2i+1}$ and hence $x_{2i-1}x_{2i+1} = \max\{x_{2i-1}, x_{2i+1}\} = x_{2i-1} \neq x_{2i+1}$ according to Lemma 7.3. $\square$

Now we are ready to complete the proof of Theorem 7.1. Five cases are possible.

- 1) $2m = n$. In this case $x_k \cdots x_n = x_k \cdots x_{2m}$ is a $V_{\pm}$ -word by Lemmas 7.6 and 7.7.
- 2) $2m + 1 = n$ and $k = 2m$. In this case $x_k \cdots x_n = x_{2m}x_{2m+1}$ is a $V_{\mp}$ -word.
- 3) $2m + 1 = n$ and $k \leq 2m - 1$. This case has three subcases.
 - 3a) If $x_{2m-1} < x_{2m+1}$, then $x_k \cdots x_n = x_k \cdots x_{2m-1}x_{2m}x_{2m+1}$ is a $V_{\mp}$ -word by Lemmas 7.6 and 7.7.
 - 3b) If $x_{2m-1} > x_{2m+1}$, then $x_k \cdots x_n = x_k \cdots x_{2m-1}x_{2m}x_{2m+1}$ is a $V_{\pm}$ -word by Lemmas 7.6 and 7.7.
 - 3c) If $x_{2m-1} = x_{2m+1}$, then $x_k \cdots x_n = x_k \cdots x_{2m-1}x_{2m}x_{2m+1}$ is a W_+ -word by Lemmas 7.6 and 7.7.
- 4) $2m + 2 \leq n$ and $k = 2m$. Since $x_{2m} \leq x_{2m+2}$, this case has two subcases.
 - 4a) If $x_{2m} < x_{2m+2}$, then $x_k \cdots x_n = x_{2m}x_{2m+1}x_{2m+2} \cdots x_n$ is a $V_{\mp}$ -word by Lemmas 7.8 and 7.9.
 - 4b) If $x_{2m} = x_{2m+2}$, then $x_k \cdots x_n = x_{2m}x_{2m+1}x_{2m+2} \cdots x_n$ is a W_- -word by Lemmas 7.8 and 7.9.
- 5) $2m + 2 \leq n$ and $k \leq 2m - 1$. This case has four subcases.
 - 5a) $x_{2m} < x_{2m+2}$ and $x_{2m-1} < x_{2m+1}$. In this case $x_k \cdots x_n = x_k \cdots x_{2m}x_{2m+1} \cdots x_n$ is a $V_{\mp}$ -word by Lemmas 7.6–7.9.

5b) $x_{2m} < x_{2m+2}$ and $x_{2m-1} > x_{2m+1}$. In this case $x_k \dots x_n = x_k \dots x_{2m-1} x_{2m} \dots x_n$ is a $V_{\pm}$ -word by Lemmas 7.6–7.9.

5b) $x_{2m} < x_{2m+2}$ and $x_{2m-1} = x_{2m+1}$. In this case $x_k \dots x_{2m-1} x_{2m} x_{2m+1} \dots x_n$ is a W_{+} -word by Lemmas 7.6–7.9.

5c) $x_{2m} = x_{2m+2}$. In this case we shall prove that $x_{2m-1} > x_{2m+1}$. Assuming that $x_{2m-1} \leq x_{2m+1}$ we can apply Lemma 7.3 to conclude that $x_{2m-1} x_{2m+1} = \max\{x_{2m-1}, x_{2m+1}\} = x_{2m-1}$. It follows from $x_{2m} = x_{2m+2}$ that $x_{2m} x_{2m+2} = x_{2m+2}$. By Lemma 7.5, the alternating word $x_{2m-1} x_{2m} x_{2m+1} x_{2m+2}$ is reducible, which is a contradiction. So, $x_{2m-1} > x_{2m+1}$ and hence $x_k \dots x_n = x_k \dots x_{2m} x_{2m+1} x_{2m+2} \dots x_n$ is a W_{-} -word by Lemmas 7.6–7.9.

Therefore each irreducible word in FS_L is a Kuratowski word, which implies that $|K| \leq |\mathcal{K}_L|$. If the set L is finite, then the set $\mathcal{K}_L$ of Kuratowski words over L has cardinality $|\mathcal{K}_L| = K(|L_{-}|, |L_{+}|)$, see Theorem 5.2. $\square$

8 Separation of Kuratowski words by homomorphisms

In the preceding section we proved that any element of a Kuratowski monoid K with a linear generating set L can be represented by a Kuratowski word $w \in \mathcal{K}_L$. In this section we shall prove that Kuratowski words can be separated by homomorphisms into the Kuratowski monoids of suitable 2-topological spaces.

Given an n -topological space $\mathbf{X} = (X, (\tau_i)_{i \in n})$, observe that the linear generating set

$$L(\mathbf{X}) = \{\check{\tau}_i\}_{i \in n} \cup \{1_X\} \cup \{\bar{\tau}_i\}_{i \in n}$$

of its Kuratowski monoid $K(\mathbf{X})$ is symmetric.

This observation motivates the following definition. A $\ast$ -linearly ordered set is a linearly ordered set L endowed with an involutive bijection $\ast : L \rightarrow L$, $\ast : \ell \mapsto \ell^{\ast}$, that has a unique fixed point $1 \in L$ and is *decreasing* in the sense that for any elements $x < y$ in L we get $x^{\ast} > y^{\ast}$. Each $\ast$ -linearly ordered set L is pointed – the unit of L is the unique fixed point of the involution $\ast : L \rightarrow L$. Observe that the structure of a $\ast$ -linearly ordered set L is determined by the structure of its negative part L_{-} .

A map $f : L \rightarrow \Lambda$ between two $\ast$ -linearly ordered sets L, Λ will be called a $\ast$ -morphism if f is *monotone* (in the sense that for any elements $x \leq y$ of L we get $f(x) \leq f(y)$) and *preserves the involution* (in the sense that $f(x^{\ast}) = f(x)^{\ast}$ for every $x \in L$). Since $f(1) = f(1^{\ast}) = f(1)^{\ast}$, the image $f(1)$ of the unit of L coincides with the unit of Λ . Observe that each $\ast$ -morphism $f : L \rightarrow \Lambda$ is uniquely determined by its restriction $f|L_{-}$.

For a $\ast$ -linearly ordered set L , the involution $\ast : L \rightarrow L$ of L has a unique extension to an involutive semigroup isomorphism $\ast : FS_L \rightarrow FS_L$ of the free semigroup over L . The image of a word $w \in FS_L$ under this involutive isomorphism will be denoted by $w^{\ast}$.

Let $\mathbf{X} = (X, \mathcal{T})$ be a polytopological space and

$$L(\mathbf{X}) = \{\check{\tau} : \tau \in \mathcal{T}\} \cup \{1_X\} \cup \{\bar{\tau} : \tau \in \mathcal{T}\}$$

be the linear generating set of the Kuratowski monoid $K(\mathbf{X})$ of $\mathbf{X}$. Observe that each topology τ is determined by its interior operator $\check{\tau}$ (since $\tau = \{\check{\tau}(A) : A \subset X\}$). This implies that the interior operators $\check{\tau}, \tau \in \mathcal{T}$, are pairwise distinct. The same is true for the closure operators $\bar{\tau}, \tau \in \mathcal{T}$. This allows us to define a bijective involution $\ast : L(\mathbf{X}) \rightarrow L(\mathbf{X})$ letting $\check{\tau}^{\ast} = \bar{\tau}$ and $\bar{\tau}^{\ast} = \check{\tau}$ for every $\tau \in \mathcal{T}$. This involution turns $L(\mathbf{X})$ into a $\ast$ -linearly ordered set.

Let L be a $\ast$ -linearly ordered set. Choose any point $c \notin L$ and consider the free semigroup $FS_{L \cup \{c\}}$ over the set $L \cup \{c\}$. This semigroup consists of words in the alphabet $L \cup \{c\}$. Let $\mathbf{X} = (X, \mathcal{T})$ be a polytopological space and $L(\mathbf{X})$ be the linear generating set of the Kuratowski monoid $K(\mathbf{X})$ of $\mathbf{X}$. Let $c_X : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $c_X : A \mapsto X \setminus A$, denote the operator of taking complement.

Given any $\ast$ -morphism $f : L \rightarrow L(\mathbf{X})$ let $\hat{f} : FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X})$ be a (unique) semigroup homomorphism such that $\hat{f}(1) = 1_X$, $\hat{f}(c) = c_X$, and $\hat{f}(\ell) = f(\ell)$ for $\ell \in L$. The homomorphism $\hat{f}$ will be called the *Kuratowski extension* of f .

Observe that $\hat{f}(\mathcal{K}_L) \subset K(\mathbf{X})$. In the semigroup $FS_{L \cup \{c\}}$ consider the subset

$$\tilde{\mathcal{K}}_L = \mathcal{K}_L \cup \{cw : w \in \mathcal{K}_L\} \subset FS_{L \cup \{c\}}$$

whose elements will be called *full Kuratowski words*.

Theorem 8.1. *For any $\star$ -linearly ordered set L and any two distinct words $u, v \in \tilde{\mathcal{K}}_L$ there is a 2-topological space $\mathbf{X}$, and a $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ whose Kuratowski extension $\hat{f} : FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X})$ separates the words u, v in the sense that $\hat{f}(u) \neq \hat{f}(v)$.*

Proof. In most of cases the underlying set of the 2-topological space $\mathbf{X}$ will be a set $X = \{x, y\}$ containing two pairwise distinct points x, y and the topologies of $\mathbf{X}$ are equal to one of four possible topologies on X :

- $\tau_d = \{\emptyset, \{x\}, \{y\}, X\}$, the discrete topology on X ;
- $\tau_a = \{\emptyset, X\}$, the anti-discrete topology on X ;
- $\tau_x = \{\emptyset, \{x\}, X\}$;
- $\tau_y = \{\emptyset, \{y\}, X\}$.

Fix any two distinct words $u, v \in \tilde{\mathcal{K}}_L$ and consider four cases.

1) $u \in \mathcal{K}_L$ and $v \notin \mathcal{K}_L$. In this case consider the 2-topological space $\mathbf{X} = (X, (\tau_a, \tau_d))$. Then for the $\star$ -morphism $f : L \rightarrow \{1_X\} \subset L(\mathbf{X})$ we get $\hat{f}(u) = 1_X \neq c_X = \hat{f}(v)$.

2) $v \in \mathcal{K}_L$ and $u \notin \mathcal{K}_L$. In this case take the 2-topological space $\mathbf{X}$ from the preceding case and observe that for the $\star$ -morphism $f : L \rightarrow \{1_X\} \subset L(\mathbf{X})$ we get $\hat{f}(u) = c_X \neq 1_X = \hat{f}(v)$.

3) $u, v \in \mathcal{K}_L$. Denote by $u_0, v_0 \in L$ the last letters of the words u, v , respectively. Consider two cases.

3a) $u_0 \neq v_0$. We lose no generality assuming that $u_0 < v_0$ (in the linearly ordered set L).

Five subcases are possible:

3aa) $u_0 = 1$ and $v_0 \in L_+$. In this case consider the 2-topological space $\mathbf{X} = (\{x, y\}, (\tau_a, \tau_d))$ and the $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to each $\ell \in L_-$ the operator $\check{\tau}_a$. Then for the subset $A = \{x\}$ of $\mathbf{X}$ and the operators $\hat{u} = \hat{f}(u)$ and $\hat{v} = \hat{f}(v)$, we get $\hat{u}(A) = A \neq X = \hat{v}(A)$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

3ab) $u_0 \in L_-$ and $v_0 = 1$. In this case, take the 2-topological space $\mathbf{X}$, the subset $A = \{x\}$, and the $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ from the preceding case. Then $\hat{u}(A) = \emptyset \neq A = \hat{v}(A)$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

3ac) $u_0 \in L_-$ and $v_0 \in L_+$. In this case, take the 2-topological space $\mathbf{X}$, the subset $A = \{x\}$ and the $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ from case (3aa). Then $\hat{u}(A) = \emptyset \neq X = \hat{v}(A)$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

3ad) $u_0, v_0 \in L_-$. Consider the 2-topological space $\mathbf{X} = (\{x, y\}, (\tau_a, \tau_x))$ and the $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to each $\ell \in L_-$ the operator

$$f(\ell) = \begin{cases} \check{\tau}_a & \text{if } \ell \leq u_0; \\ \check{\tau}_x & \text{if } \ell > u_0. \end{cases}$$

Put $\hat{u} = \hat{f}(u)$ and $\hat{v} = \hat{f}(v)$. Observe that for the subset $A = \{x\}$ we get $\hat{u}_0(A) = \check{\tau}_a(A) = \emptyset$ and hence $\hat{u}(A) = \emptyset$. Next, we evaluate $\hat{v}(A)$. Write the Kuratowski word v as $v = v_q \dots v_0$ where $v_0, \dots, v_q \in L \setminus \{1\}$. If $q = 0$, then $\hat{v}(A) = \hat{v}_0(A) = \check{\tau}_x(A) = A \neq \emptyset = \hat{u}(A)$. If $q > 0$, then $\hat{v}_1 \hat{v}_0(A) = \hat{v}_1(A) \in \{\check{\tau}_a(A), \check{\tau}_x(A)\} = \{X\}$ and hence $\hat{v}(A) = X \neq \emptyset = \hat{u}(A)$. This yields the desired inequality $\hat{f}(u) \neq \hat{f}(v)$.

3ae) $u_0, v_0 \in L_+$. In this case we can consider the conjugated words u^* and v^* and observe that their last letters are distinct and belong to the set L_- . By the preceding item, there are a 2-topological space $\mathbf{X}$ and a $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ such that $\hat{f}(u^*) \neq \hat{f}(v^*)$. Then $\hat{f}(u)^* = \hat{f}(u^*) \neq \hat{f}(v^*) = \hat{f}(v)^*$ and hence $\hat{f}(u) \neq \hat{f}(v)$.

Next, consider the case:

3b) $u_0 = v_0$. It follows that $u_0 = v_0 \in L \setminus \{1\}$. Write the Kuratowski words u, v as $u = u_p \dots u_0$ and $v = v_q \dots v_0$ where $u_i, v_i \in L \setminus \{1\}$. For the sake of consistency it will be convenient to assume that $u_i = 1$ for every integer $i \notin \{0, \dots, p\}$ and $v_j = 1$ for every integer $j \notin \{0, \dots, q\}$. In particular, $u_{-1} = v_{-1} = 1$.

Since $u \neq v$, the number $k = \min\{i \geq 0 : u_i \neq v_i\}$ is well-defined and does not exceed $\max\{p, q\}$. It follows from $u_0 = v_0$ that $k \geq 1$. By the definition of k we also get the equality $u_{k-1} \dots u_0 = v_{k-1} \dots v_0$. First we consider the case $u_{k-1} = v_{k-1} \in L_+$. Since $u_k \neq v_k$, we lose no generality assuming that $u_k < v_k$. Since u, v are alternating words, the inclusion $u_{k-1} = v_{k-1} \in L_+$ implies $u_k, v_k \in L_- \cup \{1\}$ and $u_k \in L_-$.

Two cases are possible.

3ba) $u_k < u_{k-2}$ (this case includes also the case of $k = 1$ in which $u_1 < 1 = u_{-1}$). Since $u \in \mathcal{K}_L$, the inequality $u_k < u_{k-2}$ implies that the sequence $(u_{k-2i})_{0 \leq i \leq \frac{k}{2}}$ is strictly increasing in L_- and the sequence $(u_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}} = (v_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}}$ is a strictly decreasing in L_+ .

Now consider two subcases.

3baa) $v_{k+1} > v_{k-1}$. In this case $(v_{k+1-2i}^*)_{0 \leq i \leq \frac{k+1}{2}}$ is a strictly increasing sequence in L_- . Depending on the relation between the elements u_k and v_{k+1}^* of L_- we shall distinguish two subsubcases.

3baaa) $v_{k+1}^* \leq u_k$. In this case consider the 2-topological space $\mathbf{X} = (X, (\tau_a, \tau_y))$ where $X = \{x, y\}$ and define the $*$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to each $\ell \in L_-$ the operator

$$f(\ell) = \begin{cases} \check{\tau}_a & \text{if } \ell \leq v_{k+1}^*; \\ \check{\tau}_y & \text{if } v_{k+1}^* < \ell \leq u_k; \\ 1_X & \text{if } u_k < \ell. \end{cases}$$

Consider the subset $A = \{x\} \subset X$. Since the sequence $(u_{k-2i})_{0 \leq i \leq \frac{k}{2}}$ is strictly increasing in L_- , for every positive number $i \leq \frac{k}{2}$ we get $v_{k-2i} = u_{k-2i} > u_k$ and hence $\hat{u}_{k-2i} = \hat{v}_{k-2i} = 1_X$, which implies that $\hat{u}_{k-2i}(A) = \hat{v}_{k-2i}(A) = A$. On the other hand, for every positive $i \leq \frac{k+1}{2}$ we get $u_{k+1-2i}^* = v_{k+1-2i}^* > v_{k+1}^*$ and hence $\hat{u}_{k+1-2i} = \hat{v}_{k+1-2i} \in \{\check{\tau}_y, 1_X\}$, which implies $\hat{u}_{k+1-2i}(A) = \hat{v}_{k+1-2i}(A) \in \{\check{\tau}_y(A), 1_X(A)\} = \{A\}$. So, $\hat{u}_{k-1} \dots \hat{u}_0(A) = \hat{v}_{k-1} \dots \hat{v}_0(A) = A$.

Observe that $\hat{u}_k \dots \hat{u}_1(A) = \hat{u}_k(A) = \check{\tau}_y(A) = \emptyset$ and hence $\hat{u}(A) = \emptyset$. On the other hand, the inequality $v_k > u_k$ implies that $\hat{v}_k = 1_X$ and then $\hat{v}_k \dots \hat{v}_0(A) = \hat{v}_k(A) = A$ and $\hat{v}_{k+1} \dots \hat{v}_0(A) = \hat{v}_{k+1}(A) = \check{\tau}_a(A) = X$ and then $\hat{v}(A) = X \neq \emptyset = \hat{u}(A)$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

3baab) $v_{k+1}^* > u_k$. In this case consider the 2-topological space $\mathbf{X} = (X, (\tau_a, \tau_x))$ where $X = \{x, y\}$ and define a $*$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to each $\ell \in L_-$ the operator

$$f(\ell) = \begin{cases} \check{\tau}_a & \text{if } \ell \leq u_k; \\ \check{\tau}_x & \text{if } u_k < \ell \leq v_{k+1}^*; \\ 1_X & \text{if } v_{k+1}^* < \ell. \end{cases}$$

Consider the subset $A = \{x\} \subset X$. Since the sequence $(u_{k-2i})_{0 \leq i \leq \frac{k}{2}}$ is strictly increasing in L_- , for every positive $i \leq \frac{k}{2}$ we get $v_{k-2i} = u_{k-2i} > u_k$ and hence $\hat{u}_{k-2i} = \hat{v}_{k-2i} \in \{\check{\tau}_x, 1_X\}$, which implies that $\hat{u}_{k-2i}(A) = \hat{v}_{k-2i}(A) \in \{\check{\tau}_x(A), 1_X(A)\} = \{A\}$. Since the sequence $(v_{k+1-2i}^*)_{0 \leq i \leq \frac{k+1}{2}}$ is strictly increasing in L_- , for every positive $i \leq \frac{k+1}{2}$ we get $u_{k+1-2i}^* = v_{k+1-2i}^* > v_{k+1}^*$ and hence $\hat{u}_{k+1-2i} = \hat{v}_{k+1-2i} = 1_X$, which implies $\hat{u}_{k+1-2i}(A) = \hat{v}_{k+1-2i}(A) = A$. So, $\hat{u}_{k-1} \dots \hat{u}_0(A) = \hat{v}_{k-1} \dots \hat{v}_0(A) = A$.

Observe that $\hat{u}_k \dots \hat{u}_1(A) = \hat{u}_k(A) = \check{\tau}_a(A) = \emptyset$ and hence $\hat{u}(A) = \emptyset$. Next, we evaluate $\hat{v}(A)$. If $v_k = 1$, then $\hat{v}(A) = \hat{v}_{k-1} \dots \hat{v}_0(A) = A \neq \emptyset = \hat{u}(A)$. If $v_k \neq 1$ but $v_{k+1} = 1$, then $u_k < v_k$ implies that $\hat{v}_k \in \{\check{\tau}_x, 1_X\}$ and hence $\hat{v}(A) = \hat{v}_k \dots \hat{v}_0(A) = \hat{v}_k(A) \in \{\check{\tau}_x(A), 1_X(A)\} = \{A\}$ and hence $\hat{v}(A) = A \neq \emptyset = \hat{u}(A)$. If $v_{k+1} \neq 1$, then $\hat{v}_{k+1} \dots \hat{v}_1(A) = \hat{v}_{k+1}(A) = \check{\tau}_x(A) = X$ and then $\hat{v}(A) = X \neq \emptyset = \hat{u}(A)$. This yields the desired inequality $\hat{f}(u) \neq \hat{f}(v)$.

Next, consider the subcase:

3bab) $v_{k+1} = v_{k-1}$. In this case $v \in \mathcal{W}_+$ and hence the sequences $(v_{k-2i})_{0 \leq i \leq \frac{k}{2}}$ and $(v_{k+2i})_{0 \leq i \leq \frac{q-k}{2}}$ are strictly increasing in L_- whereas the sequences $(v_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}}$ and $(v_{k+1+2i})_{0 \leq i \leq \frac{q-k-1}{2}}$ are strictly decreasing in L_+ . Consider the 2-topological space $\mathbf{X} = (\{x, y\}, (\tau_a, \tau_y))$ and define a $*$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to each element $\ell \in L_-$ the operator

$$f(\ell) = \begin{cases} \check{\tau}_y & \text{if } \ell \leq u_k; \\ 1_X & \text{if } \ell > u_k. \end{cases}$$

Also consider the subset $A = \{x\}$ in the 2-topological space $\mathbf{X}$.

Taking into account that $u_k < v_k \leq v_{k+2i}$ for any $i \in \mathbb{Z}$ with $0 \leq k+2i \leq q$, we conclude that $\hat{v}_{k+2i} = 1_X$. On the other hand, for every $i \in \mathbb{Z}$ with $1 \leq k+1+2i \leq q$ we get $\hat{v}_{k+1+2i} \in \{\bar{\tau}_y, 1_X\}$ and hence $\hat{v}_{k+1+2i}(A) \in \{\bar{\tau}_y(A), \bar{\tau}_d(A)\} = \{A\}$. This implies that $\hat{v}(A) = A$.

On the other hand, $\hat{u}_k \cdots \hat{u}_0(A) = \hat{u}_k \hat{v}_{k-1} \cdots \hat{v}_0(A) = \hat{u}_k(A) = \bar{\tau}_y(A) = \emptyset$ and then $\hat{u}(A) = \emptyset \neq A = \hat{v}(A)$. This implies that $\hat{f}(u) \neq \hat{f}(v)$.

Finally, consider the subcase:

3bac) $v_{k+1} < v_{k-1}$. Taking into account that $v \in \mathcal{K}_L$ and the sequence $(v_{k-1-2i})_{0 \leq i \leq \frac{k-1}{2}}$ is strictly decreasing in L_+ , we conclude that $v_{k-1} > v_{k-1+2i}$ for any non-zero integer number i with $0 \leq k-1+2i \leq q$ and $u_k < \min\{u_{k-2}, v_k\} = \min\{v_{k-2}, v_k\} \leq v_{k+2i}$ for any integer number i with $k+2i \in \{0, \dots, q\}$. Take the 2-topological space $\mathbf{X}$, the subset $A = \{x\}$, and the $\ast$ -morphism $f : L \rightarrow L(\mathbf{X})$ from the case (3bab). By analogy with the preceding case it can be shown that $\hat{v}(A) = A \neq \emptyset = \hat{u}(A)$ and hence $\hat{f}(u) \neq \hat{f}(v)$. This completes the proof of case (3ba).

So, now we consider the case:

3bb) $u_k \geq u_{k-2}$. This case is more difficult and requires to consider a set $X = \{x, y, z\}$ of cardinality $|X| = 3$ endowed with the topologies:

- $\tau_{x,z} = \{\emptyset, \{x\}, \{z\}, \{x, z\}, X\}$,
- $\tau_{x,z,yz} = \{\emptyset, \{x\}, \{z\}, \{y, z\}, \{x, z\}, X\}$,
- $\tau_{x,z,xy} = \{\emptyset, \{x\}, \{z\}, \{x, y\}, \{x, z\}, X\}$.

In X consider the subset $A = \{x\}$.

The inequality $u_{k-2} \leq u_k \in L_-$ and the inclusion $u \in \mathcal{K}_L$ imply that the sequence $(u_{k-1+2i})_{0 \leq i \leq \frac{p-k+1}{2}}$ is strictly decreasing in L_+ . By analogy, the (strict) inequality $v_{k-2} = u_{k-2} \leq u_k < v_k$ implies that sequence $(v_{k-2+2i})_{0 \leq i \leq \frac{q-k+2}{2}}$ is strictly increasing in L_- and $(v_{k-1+2i})_{0 \leq i \leq \frac{q-k+1}{2}}$ is strictly decreasing in L_+ . Let u_{k-1}^* be the letter in L_- , symmetric to the letter u_{k-1} with respect to 1. Depending on the relation between u_{k-1}^* and u_k we consider two subcases:

3bba) $u_{k-1}^* \leq u_k$. In this case consider the 2-topological space $\mathbf{X} = (X, (\tau_{x,z}, \tau_{x,z,yz}))$ where $X = \{x, y, z\}$ and define the $\ast$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to every element $\ell \in L_-$ the operator

$$f(\ell) = \hat{\ell} := \begin{cases} \bar{\tau}_{x,z} & \text{if } \ell \leq u_{k-1}^*; \\ \bar{\tau}_{x,z,yz} & \text{if } u_{k-1}^* < \ell \leq u_k; \\ 1_X & \text{if } u_k < \ell. \end{cases}$$

Consider the subset $A = \{x\}$ of X . Observe that for every $\ell \in L$ we get $\{x\} \subset \hat{\ell}(\{x\}) \subset \hat{\ell}(\{x, y\}) \subset \{x, y\}$. Then $\{x\} \subset \hat{u}_{k-2} \hat{u}_{k-3} \cdots \hat{u}_0(A) \subset \{x, y\}$ and

$$\{x, y\} = \bar{\tau}_{x,z}(\{x\}) = \hat{u}_{k-1}(\{x\}) \subset \hat{u}_{k-1} \hat{u}_{k-2} \cdots \hat{u}_0(A) \subset \hat{u}_{k-1}(\{x, y\}) = \bar{\tau}_{x,z}(\{x, y\}) = \{x, y\}.$$

So, $\hat{v}_{k-1} \cdots \hat{v}_0(A) = \hat{u}_{k-1} \cdots \hat{u}_0(A) = \{x, y\}$ and $\hat{u}_k \cdots \hat{u}_1(A) = \hat{u}_k(\{x, y\}) = \bar{\tau}_{x,z,yz}(\{x, y\}) = \{x\} = A$.

Since the sequence $(u_{k-1+2i})_{0 \leq i \leq \frac{p-k+1}{2}}$ is strictly decreasing in L_+ , the dual sequence $(u_{k-1+2i}^*)_{0 \leq i \leq \frac{p-k+1}{2}}$ is strictly increasing in L_- . Then $\{\hat{u}_{k-1+2i}\}_{1 \leq i \leq \frac{p-k-1}{2}} \subset \{\bar{\tau}_{x,z,yz}, 1_X\}$ and then $\hat{u}(A) = \hat{u}_p \cdots \hat{u}_{k+1}(A) \subset A$.

On the other hand, $v_k > u_k$ implies that $\hat{v}_k \cdots \hat{v}_1(A) = \hat{v}_k(\{x, y\}) = 1_X(\{x, y\}) = \{x, y\}$. Taking into account that $u_k < v_k$ and the sequence $(v_{k+2i})_{0 \leq i \leq \frac{q-k}{2}}$ is strictly increasing in L_- , we conclude that $\{v_{k+2i}\}_{0 \leq i \leq \frac{q-k}{2}} \subset \{1_X\}$, which implies that $\hat{v}(A) = \hat{v}_k \cdots \hat{v}_0(A) \supset \{x, y\} \neq \{x\} \supset \hat{u}(A)$. This yields the desired inequality $\hat{f}(u) \neq \hat{f}(v)$.

3bbb) $u_k < u_{k-1}^*$. In this case consider the 2-topological space $\mathbf{X} = (X, (\tau_{x,z}, \tau_{x,z,xy}))$ where $X = \{x, y, z\}$. Define a $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ assigning to every element $\ell \in L_-$ the operator

$$f(\ell) = \begin{cases} \check{\tau}_{x,z} & \text{if } \ell \leq u_k; \\ \check{\tau}_{x,z,xy} & \text{if } u_k < \ell \leq u_{k-1}^*; \\ 1_X & \text{if } u_{k-1}^* < \ell. \end{cases}$$

Consider the subset $A = \{x\}$ of X . It follows that

$$\hat{u}_{k-1} \cdots \hat{u}_0(A) \in \{\hat{u}_{k-1}(\{x\}), \hat{u}_{k-1}(\{x, y\})\} = \{\check{\tau}_{x,z,xy}(\{x\}), \check{\tau}_{x,z,xy}(\{x, y\})\} = \{\{x, y\}\}$$

and hence $\hat{u}_k \cdots \hat{u}_0(A) = \hat{u}_k(\{x, y\}) = \check{\tau}_{x,z}(\{x, y\}) = \{x\}$. Taking into account that the sequence $(u_{k-1+2i}^*)_{0 \leq i \leq \frac{p-k+1}{2}}$ is strictly increasing in L_- , we conclude that $\hat{u}_{k-1+2i} = 1_X$ for all positive $i \leq \frac{p-k+1}{2}$. This implies that $\hat{u}(A) = \hat{u}_p \cdots \hat{u}_{k+1}(A) \subset A = \{x\}$.

On the other hand, $\hat{v}_k \cdots \hat{v}_0(A) = \hat{v}_k(\{x, y\}) \in \{\check{\tau}_{x,z,xy}(\{x, y\}), 1_X(\{x, y\})\} = \{\{x, y\}\}$. Taking into account that $u_k < v_k$ and the sequence $(v_{k+2i})_{0 \leq i \leq \frac{q-k}{2}}$ is strictly increasing, we conclude that $\{\hat{v}_{k+2i}\}_{0 \leq i \leq \frac{q-k}{2}} \subset \{\check{\tau}_{x,z,xy}, 1_X\}$, which implies that $\hat{v}(A) = \hat{v}_q \cdots \hat{v}_{k+1}(\{x, y\}) \supset \{x, y\}$. So, $\hat{u}(A) \neq \hat{v}(A)$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

This completes the proof of case (3) under the assumption $u_{k-1} = v_{k-1} \in L_+$. If $u_{k-1} = v_{k-1} \in L_-$, then we can consider the dual words $u^* = u_p^* \cdots u_0^*$ and $v^* = v_q^* \cdots v_0^*$. For these dual words we get $u_i^* = v_i^*$ for $i < k$ and $v_{k-1}^* = u_{k-1}^* \in L_+$. In this case the preceding proof yields an 2-topological space $\mathbf{X}$ and a $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ such that $\hat{f}(u)^* = \hat{f}(u^*) \neq \hat{f}(v^*) = \hat{f}(v)^*$, which implies that $\hat{f}(u) \neq \hat{f}(v)$.

Finally, consider the case:

4) $u, v \in \tilde{\mathcal{K}}_L \setminus \mathcal{K}_L$. Then $u = cu'$ and $v = cv'$ for some distinct words $u', v' \in \mathcal{K}_L$. By the case (3), we can find a 2-topological space $\mathbf{X}$ and a $\star$ -morphism $f : L \rightarrow L(\mathbf{X})$ such that $\hat{f}(u') \neq \hat{f}(v')$. This means that $\hat{f}(u')(A) \neq \hat{f}(v')(A)$ for some subset $A \subset X$. Then

$$\hat{f}(u)(A) = \hat{f}(cu')(A) = \hat{f}(c)\hat{f}(u')(A) = c_X(\hat{f}(u')(A)) \neq c_X(\hat{f}(v')(A)) = \hat{f}(cv')(A) = \hat{f}(v)(A),$$

which means that $\hat{f}(u) \neq \hat{f}(v)$. □

The following corollary of Theorem 8.1 implies Theorem 1.2 and shows that the upper bound in Theorem 1.1 is exact.

Corollary 8.2. *For any $\star$ -linearly ordered set L there is an L_- -topological space $\mathbf{X} = (X, (\tau_\ell)_{\ell \in L_-})$ such that the unique semigroup homomorphism $\pi : FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X})$ such that $\pi(1) = 1_X$, $\pi(c) = c_X$, $\pi(\ell) = \check{\tau}_\ell$, $\pi(\ell^*) = \bar{\tau}_\ell$ for $\ell \in L_-$ maps bijectively the set $\mathcal{K}_L$ onto $K(\mathbf{X})$ and the set $\tilde{\mathcal{K}}_L$ onto $K_2(\mathbf{X})$. If the set L_- has finite cardinality n , then the Kuratowski monoid $K(\mathbf{X})$ of $\mathbf{X}$ has cardinality $|K(\mathbf{X})| = K(n, n) = K(n)$ and the full Kuratowski monoid $K_2(\mathbf{X})$ of $\mathbf{X}$ has cardinality $|K_2(\mathbf{X})| = 2 \cdot K(n, n) = 2K(n)$.*

Proof. Let $\tilde{\mathcal{K}}_L^2 \setminus \Delta = \{(u, v) \in \tilde{\mathcal{K}}_L \times \tilde{\mathcal{K}}_L : u \neq v\}$. By Theorem 8.1, for any distinct words $u, v \in \tilde{\mathcal{K}}_L$ there exist a 2-topological space $\mathbf{X}_{u,v} = (X_{u,v}, (\tau'_{u,v}, \tau''_{u,v}))$ and a $\star$ -morphism $f_{u,v} : L \rightarrow L(\mathbf{X})$ whose Kuratowski extension $\hat{f}_{u,v} : FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X}_{u,v})$ separates the words u, v in the sense that $\hat{f}(u) \neq \hat{f}(v)$. This means that $\hat{f}_{u,v}(u)(A_{u,v}) \neq \hat{f}_{u,v}(v)(A_{u,v})$ for some subset $A_{u,v} \subset X_{u,v}$. Let $\delta_{u,v}$ denote the discrete topology on the set $X_{u,v}$.

Define the L_- -topology $\tau_{u,v} : L_- \rightarrow \text{Top}(X_{u,v})$ on $X_{u,v}$ by the formula

$$\tau_{u,v}(\ell) = \begin{cases} \tau'_{u,v} & \text{if } f(\ell) = \check{\tau}'_{u,v}; \\ \tau''_{u,v} & \text{if } f(\ell) = \check{\tau}''_{u,v}; \\ \delta_{u,v} & \text{otherwise.} \end{cases}$$

Consider the L_- -topological space $\mathbf{X}_{u,v}^- = (X_{u,v}, \tau_{u,v})$ and observe that its full Kuratowski monoid coincides with the full Kuratowski monoid of the 2-topological space $\mathbf{X}_{u,v}$. Moreover, the Kuratowski extension $\hat{f}_{u,v} :$

$FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X}_{u,v}) = K_2(\mathbf{X}_{u,v}^-)$ of the morphism $f_{u,v}$ has the properties $\hat{f}_{u,v}(1) = 1_{X_{u,v}}$, $\hat{f}_{u,v}(c) = c_{X_{u,v}}$, $\hat{f}_{u,v}(\ell) = \check{\tau}_{u,v}(\ell)$, and $\hat{f}(\ell^*) = \bar{\tau}_{u,v}(\ell)$ for $\ell \in L_-$.

We lose no generality assuming that for any distinct pairs $(u, v), (u', v') \in \tilde{\mathcal{K}}_L^2 \setminus \Delta$ the sets $X_{u,v}$ and $X_{u',v'}$ are disjoint. This allows us to consider the disjoint union

$$X = \bigcup \{X_{u,v} : (u, v) \in \tilde{\mathcal{K}}_L^2 \setminus \Delta\}$$

and the subset

$$A = \bigcup \{A_{u,v} : (u, v) \in \tilde{\mathcal{K}}_L^2 \setminus \Delta\}$$

in X . For every $\ell \in L_-$ consider the topology τ_ℓ on the set X generated by the base $\bigcup \{\tau_{u,v}(\ell) : (u, v) \in \tilde{\mathcal{K}}_L^2 \setminus \Delta\}$. We claim that the L_- -topological space $\mathbf{X} = (X, \tau)$ (which is the direct sum of L_- -topological spaces $\mathbf{X}_{u,v}^-$) and the subset $A \subset X$ have the desired property: for any two distinct words $u, v \in \tilde{\mathcal{K}}_L$ we get $\hat{u}(A) \neq \hat{v}(A)$, where $\hat{u}$ and $\hat{v}$ are the images of u and v under the (unique) semigroup homomorphism $\pi : FS_{L \cup \{c\}} \rightarrow K_2(\mathbf{X})$ such that $\pi(1) = 1_X$, $\pi(c) = c_X$, $\pi(\ell) = \check{\tau}_\ell$, $\pi(\ell^*) = \bar{\tau}_\ell$ for $\ell \in L_-$. This follows from the fact that $\hat{u}(A) \cap X_{u,v} = \hat{f}_{u,v}(u)(A_{u,v}) \neq \hat{f}_{u,v}(v)(A_{u,v}) = \hat{v}(A) \cap X_{u,v}$.

This means that the restriction $\pi : \tilde{\mathcal{K}}_L \rightarrow K_2(\mathbf{X})$ is injective. By Theorem 7.1, $\pi(\tilde{\mathcal{K}}_L) = K(\mathbf{X})$. Since $K_2(\mathbf{X}) = K(\mathbf{X}) \cup \{c_X \circ w : w \in K(\mathbf{X})\}$, we get also that $\pi(\tilde{\mathcal{K}}_L) = K_2(\mathbf{X})$. This means that the homomorphism π maps bijectively the set $\tilde{\mathcal{K}}_L$ onto $K(\mathbf{X})$ and the set $\tilde{\mathcal{K}}_L$ onto $K_2(\mathbf{X})$.

If the set L_- has finite cardinality n , then the set $\tilde{\mathcal{K}}_L$ has cardinality $|\tilde{\mathcal{K}}_L| = K(n, n) = K(n)$ (according to Theorem 5.2) and hence $|K(\mathbf{X})| = |\tilde{\mathcal{K}}_L| = K(n, n)$ and $|K_2(\mathbf{X})| = |\tilde{\mathcal{K}}_L| = 2 \cdot K(n, n) = 2 \cdot K(n)$. $\square$

9 Free Kuratowski monoids

In this section we shall discuss free Kuratowski monoids over pointed linearly ordered sets. By a *pointed linearly ordered set* we understand a linearly ordered set $(L, \leq)$ with a distinguished point $1 \in L$ called the *unit* of L . The subsets $L_- = \{x \in L : x < 1\}$ and $L_+ = \{x \in L : 1 < x\}$ are called the *negative* and *positive* parts of L , respectively. A function $f : L \rightarrow \Lambda$ between two pointed linearly ordered sets is called a *morphism* if $f(1) = 1$ and f is monotone in the sense that $f(x) \leq f(y)$ for any elements $x \leq y$ of L .

Each pointed linearly ordered set L determines the *free Kuratowski monoid* FK_L defined as follows. On the free semigroup $FS_L = \bigcup_{n=1}^{\infty} L^n$ consider the smallest compatible partial preorder $\preceq$ extending the linear order $\leq$ of the set $L = L^1 \subset FS_L$ and containing the pairs $(x, x1)$, $(x, 1x)$, $(x1, x)$, $(1x, x)$, (x, x^2) , and (x^2, x) for $x \in L$. The compatible partial preorder $\preceq$ generates the congruence $\rho_{\preceq} = \{(v, w) \in FS_L : v \preceq w, w \preceq v\}$ on FS_L identifying the words $x, x1, 1x, x^2$ for any $x \in L$. The quotient semigroup $FK_L = FS_L / \rho_{\preceq}$ endowed with the quotient partial order is called the *free Kuratowski monoid* generated by the pointed linearly ordered set L . By $q_L : FS_L \rightarrow FK_L$ we shall denote the (monotone) quotient homomorphism. The restriction $\eta_L = q_L|_L : L \rightarrow FK_L$ is called the canonical embedding of the pointed linearly ordered set L into its free Kuratowski monoid. In Proposition 9.2 we shall see that η is indeed injective.

First we show that the free Kuratowski monoid FK_L is free in the categorical sense.

Proposition 9.1. *For any pointed linearly ordered set L and any Kuratowski monoid K with a linear generating set Λ any morphism $f : L \rightarrow \Lambda$ determines a unique monotone semigroup homomorphism $\tilde{f} : FK_L \rightarrow K$ such that $\tilde{f} \circ \eta = f$.*

Proof. Let $\tilde{f} : FS_L \rightarrow K$ be the unique semigroup homomorphism extending the morphism $f : L \rightarrow \Lambda \subset K$. The partial order $\leq$ of the Kuratowski monoid K induces the compatible partial preorder $\leq$ on FS_L defined by $u \leq v$ iff $\tilde{f}(u) \leq \tilde{f}(v)$. The monotonicity of f implies that the partial preorder $\leq$ contains the linear order of the set L . Then the minimality of the partial preorder $\preceq$ implies that $\preceq \subset \leq$. This allows us to find a unique monotone semigroup homomorphism $\tilde{f} : FK_L \rightarrow K$ such that $\tilde{f} \circ q_L = \tilde{f}$ and hence $\tilde{f} \circ \eta_L = \tilde{f} \circ q_L|_L = \tilde{f}|_L = f$. $\square$

Proposition 9.2. For any pointed linearly ordered set L the quotient homomorphism $q_L : FS_L \rightarrow FK_L$ maps bijectively the set $\mathcal{K}_L$ of Kuratowski words onto FK_L .

Proof. Theorem 7.1 implies that $q_L(\mathcal{K}_L) = FK_L$. To show that $q_L|_{\mathcal{K}_L}$ is injective, we shall apply Theorem 8.2. Choose any injective morphism $e : L \rightarrow L^*$ of the pointed linearly ordered set L into a $*$ -linearly ordered set L^* . Let $\tilde{e} : FS_L \rightarrow FS_{L^*}$ be the unique semigroup homomorphism extending the map e . The injectivity of e implies the injectivity of the homomorphism $\tilde{e}$.

By Proposition 9.1, the morphism e determines a unique monotone semigroup homomorphism $\bar{e} : FK_L \rightarrow FK_{L^*}$ such that $\bar{e} \circ \eta_L = \eta_{L^*} \circ e$. By Theorem 8.2, there exists a polytopological space $\mathbf{X}$ and a $*$ -morphism $f : L^* \rightarrow L(\mathbf{X})$ whose Kuratowski extension $\hat{f} : FS_{L^*} \rightarrow K(\mathbf{X})$ maps bijectively the set $\mathcal{K}_{L^*}$ onto $K(\mathbf{X})$. By Proposition 9.1, the $*$ -morphism $f : L^* \rightarrow L(\mathbf{X}) \subset K(\mathbf{X})$ determines a (unique) monotone semigroup homomorphism $\bar{f} : FS_{L^*} \rightarrow K(\mathbf{X})$ such that $\bar{f} \circ \eta_{L^*} = f$. Thus we obtain the commutative diagram

$$\begin{array}{ccccc}
 L & \xrightarrow{e} & L^* & & \\
 \downarrow & & \downarrow & & \\
 \mathcal{K}_L & \xrightarrow{\tilde{e}|_{\mathcal{K}_L}} & \mathcal{K}_{L^*} & & \\
 \downarrow & & \downarrow & & \\
 FS_L & \xrightarrow{\tilde{e}} & FS_{L^*} & \xrightarrow{\hat{f}} & K(\mathbf{X}) \\
 \downarrow q_L & & \downarrow q_{L^*} & \searrow \bar{f} & \\
 FK_L & \xrightarrow{\bar{e}} & FK_{L^*} & \xrightarrow{\bar{f}} & K(\mathbf{X})
 \end{array}$$

in which the map $\hat{f} \circ \tilde{e}|_{\mathcal{K}_L}$ is injective. Since $\hat{f} \circ \tilde{e} = \bar{f} \circ \bar{e} \circ q_L$, the injectivity of the map $\hat{f} \circ \tilde{e}|_{\mathcal{K}_L}$ implies the injectivity of the map $q_L|_{\mathcal{K}_L}$. Since $q_L(\mathcal{K}_L) = FK_L$, the restriction $q_L|_{\mathcal{K}_L} : \mathcal{K}_L \rightarrow FK_L$ is bijective. $\square$

Now we prove that the congruence $\rho_{\preceq}$ on FS_L determining the free Kuratowski monoid can be equivalently defined in a more algebraic fashion.

Proposition 9.3. For any pointed linearly ordered set L the congruence $\rho_{\preceq}$ on the free semigroup FS_L coincides with the smallest congruence ρ on FS_L containing the pairs $(x, x1)$, $(x, 1x)$, (x, x^2) for every $x \in L$ and the pairs $(x_1y_1x_2y_2, x_1y_2)$, $(y_2x_2y_1x_1, y_2x_1)$ for any points $x_1, x_2, y_1, y_2 \in L$ with $x_1 \leq x_2 \leq 1 \leq y_1 \leq y_2$.

Proof. First we prove that $\rho \subset \rho_{\preceq}$. The definition of the partial preorder $\preceq$ implies that $\{(x, x1), (x, 1x), (x, x^2) : x \in L\} \subset \rho_{\preceq}$. Repeating the proofs of Lemma 7.5, we can show that the set

$$\{(x_1y_1x_2y_2, x_1y_2), (y_2x_2y_1x_1, y_2x_1) : x_1, x_2, y_1, y_2 \in L, x_1 \leq x_2 \leq 1 \leq y_1 \leq y_2\}$$

is contained in the congruence $\rho_{\preceq}$. Now the minimality of the congruence ρ implies that $\rho \subset \rho_{\preceq}$.

Denote by $\rho^{\natural} : FS_L \rightarrow FS_L/\rho$ and $q_L : FS_L \rightarrow FS_L/\rho_{\preceq} = FK_L$ the quotient homomorphisms. Since $\rho \subset \rho_{\preceq}$, there is a unique homomorphism $h : FS_L/\rho \rightarrow FK_L$ making the following diagram commutative:

$$\begin{array}{ccc}
 & \mathcal{K}_L & \\
 & \downarrow i & \\
 & FS_L & \\
 \rho^{\natural} \swarrow & & \searrow q_L \\
 FS_L/\rho & \xrightarrow{h} & FK_L
 \end{array}$$

In this diagram by $i : \mathcal{K}_L \rightarrow FS_L$ we denote the identity inclusion of the set $\mathcal{K}_L$ of Kuratowski words into the free semigroup FS_L . The proof of Theorem 7.1 implies that $\rho^{\natural}(\mathcal{K}_L) = FS_L/\rho$.

On the other hand, Proposition 9.2 guarantees that the restriction $q_L|_{\mathcal{K}_L} : \mathcal{K}_L \rightarrow FK_L$ is bijective. This implies that the homomorphism h is bijective and hence $\rho_{\leq} = \rho$. $\square$

The bijectivity of the restriction $q_L|_{\mathcal{K}_L} : \mathcal{K}_L \rightarrow FK_L$ and Theorem 5.2 imply:

Corollary 9.4. *For any finite pointed linearly ordered set L the free Kuratowski monoid FK_L has cardinality*

$$|FK_L| = |\mathcal{K}_L| = K(n, p) = \sum_{i=0}^n \sum_{j=0}^p \binom{i+j}{i} \binom{i+j}{j}$$

where $n = |L_-|$ and $p = |L_+|$.

Given two non-negative natural numbers n, p , fix any pointed linearly ordered set $L_{n,p}$ with $|(L_{n,p})_-| = n$ and $|(L_{n,p})_+| = p$ and denote the free Kuratowski monoid $FK_{L_{n,p}}$ by $FK_{n,p}$.

Proposition 9.5. *For any pointed linearly ordered set L and any distinct elements $x, y \in FK_L$ there is a morphism of pointed linearly ordered sets $f : L \rightarrow L_{2,2}$ such that $\bar{f}(x) \neq \bar{f}(y)$. This implies that FK_L embeds into some power of the free Kuratowski monoid $FK_{2,2}$.*

Proof. Enlarge the pointed linearly ordered set L to a $\star$ -linearly ordered set $L^\star$ and denote by $e : L \rightarrow L^\star$ the identity embedding. Let $\tilde{e} : FS_L \rightarrow FS_{L^\star}$ be the unique semigroup homomorphism extending e . It is clear that $\tilde{e}$ is an injective map.

By Proposition 9.2, the restriction $q_L|_{\mathcal{K}_L} : \mathcal{K}_L \rightarrow FK_L$ is bijective. So, we can find Kuratowski words $u, v \in \mathcal{K}_L$ such that $q_L(u) = x$ and $q_L(v) = y$. By Theorem 8.1, for the Kuratowski words $u, v \in \mathcal{K}_L \subset \mathcal{K}_{L^\star}$ there exist a 2-topological space $\mathbf{X}$ and a $\star$ -morphism $g : L^\star \rightarrow L(\mathbf{X})$ such that $\hat{g}(u) \neq \hat{g}(v)$ where $\hat{g} : FS_{L^\star} \rightarrow K(\mathbf{X})$ is the unique semigroup homomorphism extending the $\star$ -morphism g . Moreover, the proof of Theorem 8.1 guarantees that the linear generating set $L(\mathbf{X})$ of the 2-topological space $\mathbf{X}$ is isomorphic to the $\star$ -linearly ordered set $L_{2,2}$. So, there exists a (unique) bijective $\star$ -morphism $\iota : L_{2,2} \rightarrow L(\mathbf{X})$. Let $f = \iota^{-1} \circ g : L^\star \rightarrow L_{2,2}$. The commutativity of the following diagram

$$\begin{array}{ccccc}
 L & \xrightarrow{e} & L^\star & \xrightarrow{f} & L_{2,2} \xrightarrow{\iota} L(\mathbf{X}) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{K}_L & \xrightarrow{\tilde{e}|_{\mathcal{K}_L}} & \mathcal{K}_{L^\star} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 FS_L & \xrightarrow{\tilde{e}} & FS_{L^\star} & \xrightarrow{\hat{g}} & K(\mathbf{X}) \\
 \downarrow q_L & & \downarrow q_{L^\star} & \nearrow \hat{f} & \downarrow \bar{i} \\
 FK_L & \xrightarrow{\bar{e}} & FK_{L^\star} & \xrightarrow{\bar{f}} & FK_{2,2} \xrightarrow{\bar{i}} K(\mathbf{X})
 \end{array}$$

and the inequality $\hat{g} \circ \tilde{e}(u) \neq \hat{g} \circ \tilde{e}(v)$ imply that $\bar{f}(x) \neq \bar{f}(y)$. $\square$

In fact, Proposition 9.5 can be improved as follows.

Proposition 9.6. *For any pointed linearly ordered set L and any distinct elements $x, y \in FK_L$ there is a pair $(n, p) \in \{(1, 2), (2, 1)\}$ and a morphism of pointed linearly ordered sets $f : L \rightarrow L_{n,p}$ such that $\bar{f}(x) \neq \bar{f}(y)$. This implies that FK_L embeds into some power of the partially ordered monoid $FK_{1,2} \times FK_{2,1}$.*

Proof. Because of Proposition 9.5, it suffices to prove that the points of the free Kuratowski monoid $FK_{2,2}$ can be separated by monotone homomorphisms into the Kuratowski monoids $FK_{1,2}$ and $FK_{2,1}$. Write the

*-linearly ordered set $L_{2,2}$ as $L_{2,2} = \{\check{\tau}_0, \check{\tau}_1, 1, \bar{\tau}_1, \bar{\tau}_0\}$ for some elements

$$\check{\tau}_0 < \check{\tau}_1 < 1 < \bar{\tau}_1 < \bar{\tau}_0.$$

By analogy the pointed linearly ordered sets $L_{1,2}$ and $L_{2,1}$ can be written as $L_{1,2} = \{\check{\tau}, 1, \bar{\tau}_1, \bar{\tau}_0\}$ and $L_{2,1} = \{\check{\tau}_0, \check{\tau}_1, 1, \bar{\tau}\}$. Consider the four surjective monotone morphisms

$$h_{12} : L_{2,2} \rightarrow L_{1,2}, \quad h_{23} : L_{2,2} \rightarrow L_{1,2}, \quad h_{34} : L_{2,2} \rightarrow L_{2,1} \text{ and } h_{45} : L_{2,2} \rightarrow L_{2,1}$$

such that

$$h_{12}(\check{\tau}_0) = h_{12}(\check{\tau}_1) = \check{\tau}, \quad h_{23}(\check{\tau}_1) = h_{23}(1) = 1$$

and

$$h_{34}(1) = h_{34}(\bar{\tau}_1) = 1, \quad h_{45}(\bar{\tau}_1) = h_{45}(\bar{\tau}_0) = \bar{\tau}.$$

By Proposition 9.2, the free Kuratowski monoid $FK_{2,2}$ can be identified with the 63-element set $\mathcal{K}_{L_{2,2}}$ of Kuratowski words in the alphabet $L_{2,2}$:

1, $\check{\tau}_0$, $\check{\tau}_1$, $\bar{\tau}_0$, $\bar{\tau}_1$, $\check{\tau}_0\bar{\tau}_0$, $\check{\tau}_0\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_0$, $\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}_0$, $\bar{\tau}_0\check{\tau}_1$, $\bar{\tau}_1\check{\tau}_0$, $\bar{\tau}_1\check{\tau}_1$,
 $\check{\tau}_0\bar{\tau}_0\check{\tau}_0$, $\check{\tau}_0\bar{\tau}_0\check{\tau}_1$, $\check{\tau}_0\bar{\tau}_1\check{\tau}_0$, $\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_1$,
 $\bar{\tau}_0\check{\tau}_0\bar{\tau}_0$, $\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}_1\bar{\tau}_0$, $\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_1$,
 $\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$, $\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_1$,
 $\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1$, $\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_1$,
 $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1\check{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1$,
 $\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$,
 $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_0\bar{\tau}_1$,
 $\check{\tau}_1\bar{\tau}_1\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}_1\bar{\tau}_0\check{\tau}_0\bar{\tau}_0\check{\tau}_1\bar{\tau}_1$.

In the following four lists we write down the values $h_{12}(w)$, $h_{23}(w)$, $h_{34}(w)$, $h_{45}(w)$ for the Kuratowski words w from the above list. Analyzing these four lists we can see that the quadruples $(h_{12}(w), h_{45}(w), h_{23}(w), h_{34}(w))$, $w \in \mathcal{K}_{L_{2,2}}$, are pairwise distinct, which means that the elements of the free Kuratowski monoid $FK_{2,2}$ are separated by the homomorphism $(h_{12}, h_{23}, h_{34}, h_{45}) : L_{2,2} \rightarrow L_{1,2}^2 \times L_{2,1}^2$.

The homomorphism h_{12} :

1, $\check{\tau}$, $\check{\tau}$, $\bar{\tau}_0$, $\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0$, $\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0$, $\check{\tau}\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}$, $\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}$, $\bar{\tau}_1\check{\tau}$,
 $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_1\check{\tau}$, $\check{\tau}\bar{\tau}_1\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_1\check{\tau}$, $\check{\tau}\bar{\tau}_1\check{\tau}$,
 $\bar{\tau}_0\check{\tau}\bar{\tau}_0$, $\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}\bar{\tau}_0$, $\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}\bar{\tau}_1$,
 $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0$, $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0$, $\check{\tau}\bar{\tau}_1$,
 $\bar{\tau}_0\check{\tau}$, $\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$,
 $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_1\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$, $\check{\tau}\bar{\tau}_0\check{\tau}$,
 $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$,
 $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}$,
 $\check{\tau}\bar{\tau}_0\check{\tau}$, $\bar{\tau}_1\check{\tau}\bar{\tau}_0\check{\tau}\bar{\tau}_1$.

The homomorphism h_{23} :

1, $\check{t}_0$, 1, $\bar{t}_0$, $\bar{t}_1$, $\check{t}_0\bar{t}_0$, $\check{t}_0\bar{t}_1$, $\bar{t}_0$, $\bar{t}_1$, $\bar{t}_0\check{t}_0$, $\bar{t}_0$, $\bar{t}_1\check{t}_0$, $\bar{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0$, $\check{t}_0\bar{t}_1\check{t}_0$, $\check{t}_0\bar{t}_1$, $\bar{t}_0\check{t}_0$, $\bar{t}_0$, $\bar{t}_1\check{t}_0$, $\bar{t}_1$,
 $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0$, $\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_1$, $\bar{t}_1\bar{t}_0$, $\bar{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_1$,
 $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_1$, $\bar{t}_0\check{t}_0$, $\bar{t}_0$,
 $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_1$, $\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_1\check{t}_0\bar{t}_0$,
 $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_1$,
 $\bar{t}_1\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0\check{t}_0\bar{t}_0$,
 $\bar{t}_1\check{t}_0\bar{t}_0\check{t}_0\bar{t}_1$, $\bar{t}_0\check{t}_0\bar{t}_0$.

The homomorphism h_{34} :

1 $\check{t}_0$, $\check{t}_1$, $\bar{t}_0$, 1, $\check{t}_0\bar{t}_0$, $\check{t}_0$, $\check{t}_1\bar{t}_0$, $\check{t}_1$, $\bar{t}_0\check{t}_0$, $\bar{t}_0\check{t}_1$, $\check{t}_0$, $\check{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_0$, $\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_1$, $\check{t}_0$, $\check{t}_1$,
 $\bar{t}_0\check{t}_0\bar{t}_0$, $\bar{t}_0\check{t}_0$, $\bar{t}_0\check{t}_1\bar{t}_0$, $\bar{t}_0\check{t}_1$, $\check{t}_0\bar{t}_0$, $\check{t}_0$, $\check{t}_1\bar{t}_0$, $\check{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_1$, $\check{t}_0\bar{t}_0$, $\check{t}_0$,
 $\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$, $\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_1$,
 $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_1$, $\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0$, $\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_1\bar{t}_0\check{t}_0$,
 $\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$, $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0$, $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$,
 $\check{t}_0\bar{t}_0\check{t}_0$, $\check{t}_1\bar{t}_0\check{t}_0\bar{t}_0\check{t}_1$.

The homomorphism h_{45} :

1, $\check{t}_0$, $\check{t}_1$, $\bar{t}$, $\bar{t}$, $\check{t}_0\bar{t}$, $\check{t}_0\bar{t}$, $\check{t}_1\bar{t}$, $\check{t}_1\bar{t}$, $\bar{t}\check{t}_0$, $\bar{t}\check{t}_1$, $\bar{t}\check{t}_0$, $\bar{t}\check{t}_1$,
 $\check{t}_0\bar{t}\check{t}_0$, $\check{t}_0\bar{t}\check{t}_1$, $\check{t}_0\bar{t}\check{t}_0$, $\check{t}_0\bar{t}\check{t}_1$, $\check{t}_1\bar{t}\check{t}_0$, $\check{t}_1\bar{t}\check{t}_1$, $\check{t}_1\bar{t}\check{t}_0$, $\check{t}_1\bar{t}\check{t}_1$,
 $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_1\bar{t}$, $\bar{t}\check{t}_1\bar{t}$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_1\bar{t}$, $\bar{t}\check{t}_1\bar{t}$,
 $\check{t}_0\bar{t}$, $\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$,
 $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0$, $\bar{t}\check{t}_1$,
 $\check{t}_1\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\check{t}_1\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\check{t}_0\bar{t}\check{t}_1$, $\check{t}_1\bar{t}\check{t}_0$, $\check{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\check{t}_1\bar{t}\check{t}_0\bar{t}\check{t}_1$,
 $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_1\bar{t}$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}_1\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_0\bar{t}$,
 $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\check{t}_1\bar{t}\check{t}_0\bar{t}$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}\check{t}_1$,
 $\check{t}_1\bar{t}\check{t}_0\bar{t}\check{t}_1$, $\bar{t}\check{t}_0\bar{t}$.

□

Proposition 9.6 shows that the homomorphisms into free Kuratowski monoids $FK_{n,p}$ for $n + p \leq 3$ completely determine the structure of any free Kuratowski monoid.

The following diagram shows the order structure of the free Kuratowski monoids $FK_{2,1}$ with the linear generating set $L_{2,1} = \{a, b, 1, x\}$. We identify the elements of $FK_{2,1}$ with the Kuratowski words in the alphabet

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