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Ordered sets as uniformities

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Abstract: In [7] a relation between completeness of certain uniformity on ordered sets and restrictions of homeomorphisms of compactifications is described. We shall add more details here and correct one proof.

Keywords: Ordered set, uniform space, proximally fine space, compactification

MSC: 54C10, 54D35, 54E15

1 Introduction

Basic topological and uniform notions used later can be found in the books [5] and [10]. Some other notions and relations are recalled in the next subsections.

1.1 Uniform and proximity spaces

A uniform structure on a set X is regarded either as a filter of covers of X (with respect to the order of star-refinement), or as a filter $\mathcal{E}$ of relations on X (filter with respect to inclusion $\subset$). We remind that a cover $\mathcal{A}$ star-refines a cover $\mathcal{B}$ if the star $\bigcup\{A' \in \mathcal{A} : A' \cap A \neq \emptyset\}$ of any $A \in \mathcal{A}$ is a part of some $B \in \mathcal{B}$. The relations in the other approach are reflexive, $\mathcal{E}$ has a base of symmetric relations and is transitive in the sense that every $E \in \mathcal{E}$ contains a composition $F \circ F$ for some $F \in \mathcal{E}$. For a uniform space X we denote by γX the completion of X and by sX the Samuel compactification of X , i.e., a completion of the totally bounded modification of X .

A proximity p on a set X is a symmetric relation on $\exp X$ having the next properties for $A, B, C \subset X$ ($\tilde{p}$ denotes non p):

1. $A\tilde{p}\emptyset$;
2. $A \cap B \neq \emptyset \Rightarrow ApB$;
3. $Ap(B \cup C) \Leftrightarrow$ either ApB or ApC ;
4. $A\tilde{p}(X \setminus B) \Rightarrow$ there exists C such that $A\tilde{p}(X \setminus C)$, $C\tilde{p}(X \setminus B)$.

The relation $A \in B$ (B is a proximal neighborhood of A) means $A\tilde{p}(X \setminus B)$. The collection $\mathcal{U}_A$ of all sets B with $B \ni A$ is a filter of sets containing A and such that for every $B \in \mathcal{U}_A$ there exists $C \in \mathcal{U}_A$ with $C \in B$.

Every uniformity induces a proximity in a natural way (ApB if all stars of A with respect to uniform covers meet B , or $E[A] \cap B \neq \emptyset$ for all $E \in \mathcal{E}$) and every proximity is induced by a uniformity, in fact, by many uniformities in general – such uniformities are called proximally equivalent. The set of proximally equivalent uniformities has a coarsest element. Those coarse uniformities are exactly totally bounded uniformities. In fact, the categories of proximity spaces and of totally bounded uniformities are isomorphic (the isomorphism preserves underlying sets). The totally bounded modification of a uniform space X is denoted by pX .

There may but need not exist a finest uniformity inducing a given proximity. Those finest uniformities are called proximally fine. Since the present paper will mostly deal with those spaces, we shall give more

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details now. The proximally fine spaces form a coreflective subclass of uniform spaces. A uniform space X is proximally fine iff every proximally continuous map $f : X \rightarrow M$ into a uniform (or metric) space M is uniformly continuous. We should mention such a property of proximally fine spaces is not shared by all bicoreflective subclasses of the category of uniform spaces. It is shown in [11] the zero-dimensional or star-finite reflections q do not have that property (i.e., there exists a uniform space X that is finest in the class of uniform spaces having the same q -modification but there are non-uniformly continuous mappings $X \rightarrow M$ that are uniformly continuous into qM).

In 1952 it was proved by V.A. Efremovič in [4] and by V. Vilhelm and Č. Vitner in [18] that every proximally continuous map on a metrizable uniform space is uniformly continuous. The fact that it means metrizable uniform spaces are proximally fine was then explicitly stated by J.M. Smirnov in [14]. Their proofs have the same idea. The result also easily follows from Ramsey theorem $\omega \rightarrow (\omega)_2^2$. I. Ramm and A.S. Švarc in [13] generalized that result (using similar procedure) for uniform spaces having monotone bases (that is possible to show also by using Dushnik-Miller theorem $\kappa \rightarrow (\kappa, \omega)^2$). In [6] the present author proved that any product of uniform spaces with monotone bases are proximally fine (the procedure uses several factorization theorems of proximally continuous maps defined on products – it should be mentioned that A. Hager knew that procedure for separable metrizable spaces using Mazur's factorization theorem for continuous maps). Probably the first example showing a class of proximally equivalent uniformities need not have a finest member was found by M. Katětov in [17]. From that time, many other examples were given (see, e.g., [1]).

The reason why the set of proximally equivalent uniformities need not have a finest element follows from the fact that the coarsest uniformity finer than two proximally equivalent uniformities may induce a strictly finer proximity. There appeared generalized uniformities removing that disadvantage. They were defined by J.M. Smirnov in [15] by means of covers and by E.M. Alfsen, O. Njåstad in [1] using vicinities of diagonal. A generalized uniformity on a set X defined by means of covers is a system $\mathcal{C}$ of covers of X having the next properties:

1. $\mathcal{C}$ contains with each cover $\mathcal{C}$ any other cover refined by $\mathcal{C}$;
2. every cover from $\mathcal{C}$ has a star refinement in $\mathcal{C}$;
3. every two finite covers from $\mathcal{C}$ have a joint refinement in $\mathcal{C}$.

Using relations, generalized uniformity $\mathcal{E}$ has the following properties:

1. $\mathcal{E}$ has a base composed of reflexive and symmetric relations;
2. every $E \in \mathcal{E}$ contains a composition $F \circ F$ for some $F \in \mathcal{F}$;
3. For every finite family $\{A_i\}_i$ of subsets of X and any finite family $\{E_j\}_j \subset \mathcal{E}$ there exists $E \in \mathcal{E}$ such that $E[A_i] \subset \bigcap_j E_j[A_i]$ for every i .

Every generalized uniformity induces a proximity and the set of all proximally equivalent generalized uniformities has always a finest member that coincides with the finest uniformity if it exists.

Smirnov used his generalized uniformities to define proximal completeness (see [15] for more details). A proximity space (X, p) is proximally complete if the finest generalized uniformity inducing it is complete. So, if a proximity is induced by a complete uniformity, the proximity is proximally complete (not conversely). Proximal completion of a proximity space is the completion of the finest generalized uniformity inducing it. Proximal completion of a uniform space X is the proximal completion of the induced proximity and is denoted as $\gamma_p X$. It can be described as the proximity subspace $\bigcap \{\gamma Y : Y \text{ is proximally equivalent to } X\}$ of the Samuel compactification sX . In case X is proximally equivalent to a proximally fine space Z , the proximal completion of X is the completion of Z regarded as a proximity space.

In [7] it is shown the next conditions are equivalent to proximal completeness of uniform spaces X :

1. For every point $\xi \in sX \setminus X$ there exists a uniform cover of X having no extension to an open cover of $X \cup \{\xi\}$;
2. For every point $\xi \in sX \setminus X$ there exists a proximally continuous mapping on X into a complete (or a proximally complete) space M that cannot be extended to a proximally continuous mapping on $X \cup \{\xi\}$.

The proximal completeness may be transferred to Tikhonov topological spaces using their compactifications: if bX is a compactification of a Tikhonov space X , we say X is bX -complete if the proximity on X induced by bX is proximally complete. Some examples are shown in the next list. Recall that a topological space is said to be Dieudonné complete if its topology is generated by a complete uniformity (e.g., X is paracompact or realcompact).

1. For uniform spaces X with $sX = \beta X$ the notions *proximally complete*, *Dieudonné complete*, βX -complete coincide.
2. Every completely metrizable space is sX -complete.
3. A Tikhonov space X is bX -complete provided it has a complete proximally fine uniformity u and bX is finer than the Samuel compactification of (X, u) .

1.2 Ordered sets as uniform spaces

To avoid misunderstanding, we recall several concepts concerning orders. Ordered set (or partially ordered set, briefly ordered set) $(R, \leq)$ is a non-void set endowed with a reflexive, transitive and antisymmetric relation on R (the relation is usually denoted as $\leq$).

Our basic concepts concerning ordered sets are related to increasing situations and we omit the word "right" in the next terms. We use the standard notation: for $r \in R$, $\uparrow r = \{s \in R : s \geq r\}$:

- R is *directed* if for every two points $r, s \in R$ there exists $t \in R$ with $r \leq t$ and $s \leq t$;
- for an infinite regular cardinal κ , R is $< \kappa$ -*directed* if for any subset $A \subset R$ of cardinality $|A| < \kappa$, there exists $t \in R$ with $r \leq t$ for each $r \in A$;
- $A \subset R$ is *cofinal* if $A \cap \uparrow r \neq \emptyset$ for every $r \in R$;
- $A \subset R$ is *residual* if there exists $r \in R$ such that $A \supset \uparrow r$;

In case $\kappa = \omega_1$ in the second item, we speak about countably directed sets.

In the next, our ordered sets will be directed sets without the last element.

Clearly, a subset $A \subset R$ is not residual iff $R \setminus A$ is cofinal. If $A_1 \cup A_2 \cup \dots \cup A_n$ is cofinal in R then at least one A_i is cofinal in R .

A free filter $\mathcal{F}$ on a set X is considered as an ordered set $(\mathcal{F}, \supset)$.

Any ordered set $(R, \leq)$ will be regarded as a uniform space having the following base $\{\mathcal{U}_r : r \in R\}$ of uniform covers or $\{E_r : r \in R\}$ of uniform neighborhoods of the diagonal Δ_R of R :

$$\mathcal{U}_r = \{\{q\}; q \in R\} \cup \{\uparrow r\}, \quad E_r = \Delta_R \cup (\uparrow r \times \uparrow r).$$

The uniformity on R using relations will be denoted by $\mathcal{E}_R$ and that using covers by u_R . The space R is uniformly zero-dimensional and topologically discrete. The filter base $\{\uparrow r : r \in R\}$ is Cauchy in R not converging in R . In fact, it is the unique non-converging minimal Cauchy filter in R . Consequently, R is not complete but it may be proximally complete (e.g., if R is a free ultrafilter on a set). The unique ideal point in a completion of R will be denoted by ∞_R . It is a point of $sR \setminus R$ and every proximally continuous mapping f on R into any uniform space Y extends continuously to the point ∞_R with a value in sY . The space R is not proximally complete iff the value $f(\infty_R)$ belongs to Y for any complete metric space Y .

A characterization of the proximity relation p on R is easy: if $A, B \subset R$ then ApB iff either $A \cap B \neq \emptyset$ or both sets A, B are cofinal in R .

We finish our Introduction with several easy assertions special for ordered sets.

Lemma 1.1. *Proximal neighborhoods of $A \subset R$ are characterized by the following assertions:*

- if A is not cofinal in R then $A \in A$;
- if A is cofinal in R then $U \ni A$ iff there exists $r \in R$ such that $U \supset A \cup \uparrow r$.

Thus, $A \in A$ for residual sets A .

Lemma 1.2. *A map between ordered sets is uniformly continuous iff it either preserves cofinal sets or is constant on a residual set.*

Proof. Let $f : R \rightarrow S$, R, S be ordered sets. If f is uniformly continuous then for any $s \in S$ there exists $r \in R$ such that $\mathcal{U}_r$ refines $f^{-1}(\mathcal{U}_s)$. Thus either $|f(\uparrow r)| = 1$ (then f is constant on $\uparrow r$) or all $f^{-1}(\uparrow s)$ are residual in R . The latter property means f preserves cofinal sets (otherwise we have $f(R') \cap \uparrow s = \emptyset$ for some cofinal $R' \subset R$).

Conversely, if f is constant on a residual set, it is clearly uniformly continuous. If f preserves cofinal sets and $s \in S$ has the property that $f^{-1}(\uparrow s)$ contains no $\uparrow r$, $r \in R$, then there is a cofinal set R' in R with $f(R') \cap \uparrow s = \emptyset$. $\square$

The next result is also a special property of ordered sets.

Lemma 1.3. *A proximally continuous map on an ordered set into a uniform space is uniformly continuous provided the restriction to a cofinal subset of the domain is uniformly continuous.*

Proof. Assume that a proximally continuous map $f : R \rightarrow Y$ defined on an ordered set R is uniformly continuous on a cofinal set S in R . We may assume Y is a metric space with a metric e . For any $\varepsilon > 0$ there exists $s_0 \in S$ such that $e(u, v) < \varepsilon$ for any $u, v \in S$, $u, v \geq s_0$. If $R \setminus S$ is not cofinal, there is nothing more to prove. If $T = R \setminus S$ is cofinal in R then $e(f(S), f(T')) = 0$ for any cofinal set T' in T . There is some $t_0 \in T$ such that $e(f(t), f(S)) < \varepsilon$ for each $t \in T$, $t \geq t_0$ (otherwise we could find a cofinal set T' in T with $e(f(T'), f(S)) \geq \varepsilon$). Since the diameter of $f(S \cap \uparrow s_0)$ is at most 2ε , we get $e(f(x), f(y)) < 4\varepsilon$ for every $x, y \in R$, x, y greater than both s_0, t_0 , which proves uniform continuity of f . $\square$

Recall that a subspace X of a space Y (considered either as proximity spaces or as uniform spaces) is a retract of Y if there exists a map $Y \rightarrow X$ that is identity on X and is proximally (or uniformly, resp.) continuous.

Lemma 1.4. *Every subset S of an ordered set R is a retract of R both in proximity and uniform spaces.*

Proof. If S is not cofinal, the assertion is trivial. Assume S is cofinal. For each $r \in R$ find $s_r \in S$, $s_r \geq r$, with $s_r = r$ if $r \in S$. The map $q : R \rightarrow S$ assigning s_r to r is uniformly continuous (thus proximally continuous) since it preserves cofinality. From the same reason, also the identity map $S \rightarrow R$ is uniformly continuous. $\square$

Every retraction is a quotient. Thus, subsets of an ordered set R are both uniform and proximal quotients of R .

2 Uniformly non-complete ordered sets

A reason why our uniformities on ordered sets are studied is Proposition 4.1 in the last but one section. The only condition in the proposition is proximal non-completeness of the considered ordered set. So, we shall need special characterizations of proximal (non-)completeness of ordered sets.

Proximal non-completeness of ordered sets will be shown to be equivalent to their proximal fineness. From that reason we start with a lemma showing that proximal fineness of ordered sets is a property of cofinal subsets. That assertion is easy to show for proximal non-completeness, too, but the proof is easier for proximally fine spaces. We shall use a trivial fact that the uniformity of a cofinal subset S of the uniform space R is the uniformity of the ordered set S (non-cofinal subsets S of the uniform space R are uniformly discrete and thus proximally discrete).

Lemma 2.1. *The following conditions are equivalent for an ordered set R :*

1. R is proximally fine.
2. Every cofinal subset of R is proximally fine.
3. There exists a cofinal subset of R that is proximally fine.

Proof. The implication $1 \Rightarrow 2$ follows from the fact that coreflective classes are closed under retracts, which holds in our situation by Lemma 1.4. The implication $2 \Rightarrow 3$ is trivial.

To show $3 \Rightarrow 1$, take a cofinal subset S of R that is proximally fine. If $f : R \rightarrow (M, d)$ be a proximally continuous map into a metric space, its restriction to S is uniformly continuous so that f is uniformly continuous by Lemma 1.3. $\square$

Consequently, one may assume that R has a least element or, instead of considering linearly ordered sets we may consider infinite regular cardinals.

The next interesting (and basic in this section) relation between proximal fineness and proximal completeness for ordered sets is far from being true in all uniform spaces. We use the fact that every minimal Cauchy filter in a generalized uniformity $\mathcal{F}$ finer than and proximally equivalent to a generalized uniformity $\mathcal{E}$ is a minimal Cauchy filter in $\mathcal{E}$. Recall that a Cauchy filter $\mathcal{C}$ in a uniform space is minimal if every $C \in \mathcal{C}$ contains $C' \in \mathcal{C}$ with $C' \subseteq C$.

Proposition 2.2. *An ordered set R is proximally fine iff it is not proximally complete.*

Proof. If a uniform space X is proximally fine, it is proximally complete iff it is complete. So, if R is proximally fine, it is not proximally complete. Assume now that $(R, \mathcal{E}_R)$ is not proximally fine. There exists a strictly finer generalized uniformity $\mathcal{F}$ on R . If $\mathcal{F}$ is not complete, there is a unique non-converging minimal Cauchy filter $\mathcal{C}$ in $\mathcal{F}$. The filter $\mathcal{C}$ coincides with the corresponding Cauchy filter on $\mathcal{E}$ having for its base the collection $\{\uparrow r : r \in R\}$. Since $\mathcal{C}$ is Cauchy in $\mathcal{F}$, for every $F \in \mathcal{F}$ there exists $r \in R$ with $\uparrow r \times \uparrow r \subset F$, which implies F belongs to $\mathcal{E}_r$ and that contradicts our assumption $\mathcal{F} \neq \mathcal{E}_R$. Consequently, $\mathcal{F}$ is complete and, thus, R is proximally complete. $\square$

We may now combine proximal fineness and proximal non-completeness according what is more convenient for a given situation.

If a uniform space X is not proximally fine there exists a strictly finer complete generalized uniform space Y proximally equivalent to X . In general, Y need not be a uniform space. There are topologically discrete uniformities that are not proximally fine but are proximally minimal (i.e., every strictly finer uniformity induces a strictly finer proximity) – see [6]. That cannot happen if X is an ordered set considered as a uniform space.

Proposition 2.3. *An ordered set R is proximally fine iff there is no strictly finer uniformity proximally equivalent to R .*

Proof. It suffice to show the implication $\Leftarrow$. If R is not proximally fine, there is a strictly finer complete generalized uniformity $\mathcal{F}$ proximally equivalent to $\mathcal{E}_R$. Then $\mathcal{F}$ contains a decreasing sequence $\{F_n\}_\omega$ such that $F_n \circ F_n \subset F_{n-1}$, $F_n = F_n^{-1}$ and $\uparrow r \setminus F_n \neq \emptyset$ for $n \in \omega$, $r \in R$. We show that the collection $\mathcal{F}' = \{E_r \cap F_n : r \in R, n \in \omega\}$ is a base of a uniformity on R , strictly finer than $\mathcal{E}_R$ and proximally equivalent to R .

Clearly, $\mathcal{F}'$ is a base of a filter in $R \times R$ composed of symmetric sets containing the diagonal of R . To show that it is a base for a uniformity, one must prove that for each r, n there exists s, k such that $(F_k \cap E_s) \circ (F_k \cap E_s) \subset F_n \cup E_r$. It suffices to take $s = r, k = n + 1$. It is also easy to see that $\mathcal{F}'$ is strictly finer than $\mathcal{E}$. It remains to prove that $\mathcal{F}'$ induces the proximity of R , i.e. for every $S \subset R$ and every r, n we have $(F_n \cap E_r)[S] \supseteq S$. The only nontrivial case is the situation for S cofinal and not residual in R , where we must find $t \in R, t \geq r$, such that $(F_n \cap E_r)[S] \supseteq \uparrow t$ (i.e., for every $u \geq t$ there exists $s \in S, s \geq r$, with $u \in F_n[s]$). Since $\mathcal{F}$ induces the proximity of R , there exists $t > r$ with $F_n[S \cap \uparrow r] \supseteq \uparrow t$. The last inclusion means that for every $u > t$ there exists $s \in S, s > r$, with $u \in F_n[s]$, which is equivalent to $(F_n \cap E_r)[S] \supseteq \uparrow t$. $\square$

We have now three possibilities to check whether an ordered set is proximally fine. The first one is to use the previous Proposition 2.3 and to show that no proximally equivalent uniformity is strictly finer than R . That is not difficult for linearly ordered sets but seems to be complicated for other situations.

The second possibility is to show that every proximally continuous map $f : R \rightarrow (M, d)$, where (M, d) is a metric space, is uniformly continuous. Assuming it is not the case, there exist adjacent nets $\{x_i\}_I, \{y_i\}_I$ in

R such that $d(f(x_i), f(y_i)) > p$ for some $p > 0$ and each $i \in I$. That situation may lead to contradiction with proximal continuity of f . Recall that in our case, the property for the nets being adjacent means that for every $r \in R$ there exists i_0 such that $x_i, y_i \geq r$ for each $i > i_0$ (thus the sets $\{x_i\}_I, \{y_i\}_I$ are cofinal in R). That is, probably, the most efficient method, at least for products investigated in the next section.

The last possibility follows from proximal non-completeness. For every proximally continuous map $f : R \rightarrow (M, d)$, where (M, d) is a complete metric space, the f -image of the non-converging minimal Cauchy filter in R must converge in M . If it does not converge, it has no accumulation point (see Lemma 3.2) and one can find an open cover $\mathcal{U}$ of M such that every $f^{-1}(U)$, $U \in \mathcal{U}$, is disjoint with some $\uparrow r$. That approach is interesting because it leads to easier proofs in some situations (see Proposition 3.3).

3 Products

As mentioned in Introduction, uniform spaces having linearly ordered bases are proximally fine. Since linearly ordered sets have a linearly ordered base of its uniformity, they are proximally fine. Since also any product of uniform spaces having linearly ordered bases is proximally fine we are tempted to say the same holds for products of ordered sets. A problem is that products in the category of ordered sets induce a different uniformity than the corresponding products in the category of uniform spaces. For instance, if $R = \omega \times \omega$, then for any two disjoint infinite sets A, B in ω the sets $\{n\} \times A, \{n\} \times B$ are proximal in the uniform product of ordered spaces ω and ω but not in the ordered set R . Thus, proximal fineness of an ordered product of linearly ordered sets does not follow from proximal fineness of the uniform product of linearly ordered sets (at least not directly). The proof for products of uniform spaces uses two special factorizations (to smaller number of coordinates) of proximally continuous maps on subspaces of products. That procedure is probably impossible to use for ordered products. A problem is that subproducts are usually not cofinal in products and thus, they are proximally and uniformly discrete subspaces of the product.

Proposition 2.5 in [7] states that, specially, order products of finitely many linearly ordered sets is proximally fine. The proof uses an extension of proximally continuous maps but it is not proved that the extension exists. We shall now give another proof using combinations of approaches described in the preceding section.

In this section we deal with proximal fineness of order products $R = \prod_{i=0}^n R_i$ of finitely many linearly ordered sets. Because of Lemma 2.1 it suffice to prove our assertion for products of finitely many different infinite regular cardinals. So, let $\kappa_0 < \kappa_1 < \dots < \kappa_n$ and R be the order product $\prod_{i \leq n} \kappa_i$.

We need one property of ultrafilters, a modification of Lemma 3.13 in [8].

Lemma 3.1. *Let R be an ordered product of finitely many infinite regular cardinals $\kappa_0 < \kappa_1 < \dots < \kappa_n$ and h be an order preserving map of R into a free ultrafilter $\mathcal{X}$. Then $h(R)$ is not cofinal in $\mathcal{X}$.*

Proof. The proof goes by induction. The assertion is known to hold for $n = 0$. No free ultrafilter can have a monotone base (if $\{B_\alpha\}_\kappa$ is such a base then the disjoint sets $\bigcup\{B_\alpha \setminus B_{\alpha+1} : \alpha \text{ odd}\}, \bigcup\{B_\alpha \setminus B_{\alpha+1} : \alpha \text{ even}\}$ both belong to $\mathcal{X}$). Moreover, every strictly monotone subsystem of a free ultrafilter has cardinality smaller than cofinality of the ultrafilter.

Assume our assertion holds for all $n < k$ and that $h(R)$ is cofinal in $\mathcal{X}$ for $R = \kappa_0 \times \dots \times \kappa_k$. Since R has a cofinal set of cardinality κ_k , the same must be true for $\mathcal{X}$. Since every chain in $\mathcal{X}$ has cardinality less than κ_k , for any $x \in \kappa_0 \times \dots \times \kappa_{k-1}$ there exists $\alpha_x \in \kappa_k$ such that h is constant on $\{x\} \times \{\alpha \in \kappa_k : \alpha > \alpha_x\}$. Thus $h(R)$ has cardinality at most κ_{k-1} and there exists $\alpha_k \in \kappa_k$ such that for any $x \in \kappa_1 \times \dots \times \kappa_{k-1}$, $\gamma, \delta > \alpha_k$ we have $h(x, \gamma) = h(x, \delta)$. The restriction of h to $\kappa_0 \times \dots \times \kappa_{k-1} \times (\alpha_k, \kappa_k)$ is, in fact an order-preserving map of $R' = \kappa_0 \times \dots \times \kappa_{k-1}$ into $\mathcal{X}$ with $h(R') = h(R)$. By inductive hypothesis, $h(R')$ is not cofinal in $\mathcal{X}$. $\square$

Before we prove the main result we want to give easier proof of its special case when R is countably directed. We need the following easy result known for uniformly continuous maps (realize that proximally continuous image of a Cauchy filter need not be Cauchy).

Lemma 3.2. *Let $f : X \rightarrow Z$ be a proximally continuous map between uniform spaces X, Z and $\mathcal{C}$ be a Cauchy filter in X . Every accumulation point of $f(\mathcal{C})$ is its limit point.*

Proof. Let $z \in Z$ be an accumulation point of $f(\mathcal{C})$ that is not a limit point of $f(\mathcal{C})$. There exists an open neighborhood U of z such that $C \setminus f^{-1}(U) \neq \emptyset$ for every $C \in \mathcal{C}$. Choose an open neighborhood V of z with $V \subseteq U$. We shall select points $x_C \in C \cap f^{-1}(V), y_C \in C \setminus f^{-1}(U)$ for every $C \in \mathcal{C}$. Then the sets $\{x_C\}, \{y_C\}$ are proximal in X having non-proximal f -images – a contradiction. $\square$

Theorem 3.3. *If $\kappa_0, \dots, \kappa_n$ are uncountable regular cardinals, then their ordered product R is proximally fine.*

Proof. Let $f : R \rightarrow (M, d)$ be a proximally continuous map into a complete metric space and $\mathcal{C}$ be the unique non-converging minimal Cauchy filter in R . Suppose $f(\mathcal{C})$ has no limit point in M , thus by Lemma 3.2, it has no accumulation point in M . For every $z \in M$ we find its open neighborhood U_z disjoint with the image $f(C_z)$ of some $C_z \in \mathcal{C}$. The open cover $\{U_z\}_M$ has a refinement $\mathcal{U}$ that is a union of countably many uniformly discrete collections $\mathcal{U}_n$ of open sets.

Since R is countably directed, there exist n such that the f -preimage of the union U_n of $\mathcal{U}_n = \{U_i^n\}_I$ is cofinal in R . Define $\mathcal{X} = \{J \subset I : f^{-1}(\bigcup_J U_i^n) \text{ is cofinal in } R\}$. Then $\mathcal{X}$ is a free filter on I . Indeed, if $J, K \in \mathcal{X}$ and $J \cap K \notin \mathcal{X}$ then both $J \setminus K, K \setminus J$ belong to $\mathcal{X}$, which is impossible since $\bigcup_{J \setminus K} U_i^n, \bigcup_{K \setminus J} U_i^n$ are non-proximal sets. In fact, $\mathcal{X}$ is an ultrafilter since if a union of two sets in R is cofinal, one of them must be cofinal. To every $r \in R$ assign $h(r) = \{i \in I : f^{-1}(U_i^n) \cap \uparrow r \neq \emptyset\}$. Then $h : R \rightarrow \mathcal{X}$ is order preserving, so that by Lemma 3.1 the image $h(R)$ is not cofinal in $\mathcal{X}$, which means there exists $J \in \mathcal{X}$ such that for each $r \in R$ one can find $i_r \in I \setminus J$ with $f^{-1}(U_{i_r}^n) \cap \uparrow r \neq \emptyset$. That implies $K = \{i_r\}_R$ is disjoint with J and belongs to $\mathcal{X}$ since $f^{-1}(\bigcup_R U_{i_r}^n)$ meets every $\uparrow r$. That contradiction proves the requested result. $\square$

If cofinality of $\kappa_0 \times \dots \times \kappa_n$ is countable and $\kappa_0 < \dots < \kappa_n$ are regular cardinals, the only possibility is $\kappa_0 = \omega$. We do not know whether the above proof can be modified also for that case and we use another method using proximal non-uniform continuity. To simplify our procedure we shall formulate a part of our proof as a Lemma.

Lemma 3.4. *Let $\kappa_0 < \dots < \kappa_n$ be regular infinite cardinals and d be pseudometric on the product $R = \kappa_0 \times \dots \times \kappa_n$ such that d is proximally continuous but not uniformly continuous. Then there are a cofinal set S in $\kappa_0 \times \dots \times \kappa_{n-1}$, sets $A_q, q \in S$, cofinal in $\{q\} \times \kappa_n$ and sets $B_q \subset R$ such that $\bigcup_S B_q$ is cofinal in R and $d(A_q, B_q) > \eta$ for some $\eta > 0$ and all $q \in S$.*

Proof. There are adjacent nets $\{x_i\}_I, \{y_i\}_I$ in R with $d(x_i, y_i) > \xi$ for some $\xi > 0$ and all $i \in I$. Adjacency means that for every $r \in R$ there exists $i_0 \in I$ such that $x_i, y_i \geq r$ for each $i \geq i_0$. There is a cofinal set S in Q such that $A'_q = \{x_i\} \cap (\{q\} \times \kappa_n)$ is cofinal for each $q \in S$. Modifying the Ramm-Švarc procedure, one can find a cofinal set A''_q in A'_q such that $d(x_i, y_j) > p/3$ whenever $x_i, x_j \in A''_q$ (realize that those points y_j need not belong to $\{q\} \times \kappa_n$). We shall show a proof using Dushnik-Miller theorem $\kappa_n \rightarrow (\kappa_n, \omega)^2$. To do that, we shall arrange A'_q as $\{u_\alpha\}_{\kappa_n}$ and the corresponding points y_i to $x_i \in A'_q$ as v_α . Let a two-point subset $\{\alpha, \beta\}$ of κ_n be white if $d(u_\alpha, v_\beta) > \xi/3$ and $d(v_\alpha, u_\beta) > \xi/3$; otherwise that two-point set is black. Either there is a white-homogeneous subset S_q of cardinality κ_n or a black-homogeneous countable set $N = \{\alpha_k\}_\omega$. In the former case the set S_q is the requested one. In the latter case we have for every $k, l \in \omega$ either $d(u_{\alpha_k}, v_{\beta_l}) \leq \xi/3$ or $d(v_{\alpha_k}, u_{\beta_l}) \leq \xi/3$. There exists a triple (k, l, m) such either $d(x_{\alpha_k}, y_{\alpha_l}) \leq \xi/3$ and $d(x_{\alpha_m}, y_{\alpha_l}) \leq \xi/3$ or $d(y_{\alpha_k}, x_{\alpha_l}) \leq \xi/3$ and $d(y_{\alpha_m}, x_{\alpha_l}) \leq \xi/3$. Assume the first case occurs. Then $d(x_{\alpha_k}, x_{\alpha_m}) \leq 2\xi/3$ and, since $d(x_{\alpha_k}, y_{\alpha_k}) > \xi$, we get $d(x_{\alpha_m}, y_{\alpha_k}) > \xi/3$ and that contradicts the black color of $\{\alpha_m, \alpha_k\}$.

Now, it suffices to put $A_q = A''_q, B_q$ is the set of corresponding points y_i to $x_i \in A_q$. In our case $\eta = \xi/3$. $\square$

We shall now prove our main result. For a pseudometric d on a set we denote by $N_\varepsilon(A)$ the ε -neighborhood of A , i.e., $N_\varepsilon(A) = \bigcup_{a \in A} N_\varepsilon(a)$ is the union of ε -balls $N_\varepsilon(a)$. By d_H we mean the Hausdorff pseudometric corresponding to d .

Theorem 3.5. *Ordered products of finitely many linearly ordered sets are proximally fine.*

Proof. We shall proceed by induction on number n in the product $R = \prod_{i \leq n} \kappa_i$. The result is known to hold for $n = 0$. Assume it holds for $Q = \kappa_0 \times \cdots \times \kappa_{n-1}$. We shall use the fact that cofinality of Q is κ_{n-1} and that κ_n is $< \kappa_n$ -directed (if $\bigcup_I A_i$ is a cofinal subset of κ_n and $|I| < \kappa_n$ then at least one set A_i is cofinal in κ_n). We denote by R_q the subset $\{q\} \times \kappa_n$ for $q \in Q$. When we say that a set $A \subset R_q$ is cofinal or residual, we always mean to have that property in R_q (it never has the property in R). Every cofinal set in Q has a partition into two sets cofinal in Q .

Let d be a proximally continuous pseudometric on R that is not uniformly continuous. We shall use the notation from Lemma 3.4. For $\alpha \in \kappa_n$ denote $Z_\alpha = \{(q, \beta) : q \in Q, \beta < \alpha\}$. Assume first that for a residual (equivalently, cofinal) set of indexes α and some set $\{k_\alpha\}$ of integers, the sets $N_{1/k_\alpha}(Z_\alpha) \cap A$ are not cofinal. Therefore there exist $q_\alpha \in Q, \gamma_\alpha \in \kappa_n$ and $k_\alpha \in \omega, k_\alpha \neq 0$, such that $N_{1/k_\alpha}(Z_\alpha) \cap A \cap \uparrow(q_\alpha, \gamma_\alpha) = \emptyset$. One can find a cofinal set S in κ_n such that the assignments $\alpha \rightsquigarrow q_\alpha, \alpha \rightsquigarrow k_\alpha$ are constant on S with values q', k . We can assume $S = \kappa_n$. We shall now construct a cofinal set $T = \{t_\alpha\}_{\alpha \in \kappa_n}$ in κ_n . Let $t_0 = 0$ and assume $t_\alpha, \alpha < \gamma$, are constructed having the following properties

1. for $\alpha < \beta < \gamma$ one has $\gamma_{t_\alpha} < t_\beta$;
2. if $\alpha < \gamma, q > q'$ and A_q is cofinal then $A_q \cap \{(q, \beta) : \gamma_{t_\alpha} < \beta < t_{\alpha+1}\} \neq \emptyset$;
3. if $\beta < \gamma$ is limit then $t_\beta = \sup\{t_\alpha : \alpha < \beta\}$.

It is clear there exists an ordinal $t_\gamma \in \kappa_n$ satisfying the above three properties for $\alpha \leq \beta \leq \gamma$. Take a partition of T into two cofinal subsets T_1, T_2 and define $X_i = \{(q, \delta) \in A : q > q', \gamma_\alpha < \delta < t_{\alpha+1}, \alpha \in T_i\}, i = 1, 2$. Then both X_1, X_2 are cofinal in R and $d(X_1, X_2) > 1/k$, which contradicts the proximal continuity of f .

It remains to suppose that for a cofinal set of indexes α , the sets $N_{1/k}(Z_\alpha) \cap A$ are cofinal for all integers k (of course, it suffice to have one such α). Take a fixed k and the cofinal set C in Q such that $N_{1/k}(Z_\alpha) \cap A_q$ is cofinal for each $q \in C$. Since $|Z_\alpha| < \kappa_n$, there must exist $z_q \in Z_\alpha$ for each $q \in C$ such that $A'_q = N_{1/k}(z_q) \cap A_q$ is cofinal. Thus, $\bigcup_{C'} A'_q$ is cofinal in R for any cofinal C' in C . Take a partition $\{C_1, C_2\}$ of C into sets cofinal in Q . Because of their cofinality, $d(\bigcup_{C_1} A'_q, \bigcup_{C_2} A'_q) = 0$ and the same equality holds if the unions are over the sets $\uparrow q \cap C_1, \uparrow q \cap C_2$ resp. for any $q \in Q$. For any fixed $\varepsilon > 0$ and arbitrary $q \in Q$ there are $a_r \in \uparrow q \cap C_1, b_r \in \uparrow q \cap C_2$ and $u_r \in A'_{a_r}, v_r \in A'_{b_r}$ with $d(u_r, v_r) < \varepsilon$. Since diameters of the sets A'_q are less or equal to $2/k$, we get $d_H(A'_{a_r}, A'_{a_s}) \leq 4/k + \varepsilon$. We may choose k and ε such that $4/k + \varepsilon < \eta/3$. Consequently, $d(A'_{a_r}, B_{a_s}) > \eta/3$ for any $r \in C_1, s \in C_2$, which contradicts proximal continuity of f . The assertion of our theorem is proved. $\square$

4 Applications

As we mentioned in Introduction, non-proximal completeness implies some results on restrictions of homeomorphisms of compactifications. A basic tool for such usage is the next proposition from [7].

Proposition 4.1. *Let $\{x_r\}_R$ be a net in a uniform space X converging to ξ in sX . If R is not proximally complete, then $\xi \in \gamma_p X$.*

Proof. The map $\varphi = \{r \rightsquigarrow x_r\} : R \rightarrow X$ is proximally continuous. If R is not proximally complete, φ can be extended to a (proximally) continuous mapping on $R \cup \{\infty_R\}$ into $\gamma_p X$. Consequently, $\xi = \varphi(\infty_R) \in \gamma_p X$. $\square$

There are several ways how to use the preceding result. We show one possibility here (see [7] for more details and other possibilities).

Theorem 4.2. *Let bX and bY be homeomorphic compactifications of X, Y resp. If the proximities of X, Y inherited from bX, bY are proximally complete and filters of neighborhoods in X, Y are not proximally complete, then the restriction of the homeomorphism maps X onto Y .*

Proof. Let $h : bX \rightarrow bY$ be a homeomorphism and $x \in X$. The neighborhood filters of x in X and in bX are isomorphic as ordered sets, and they are isomorphic to the neighborhood filter of $h(x)$ in bY . Every

neighborhood U of $h(x)$ in bY meets Y and we can choose some y_u from that intersection. The net $\{y_u\}$ in Y converges to $f(x)$ and has for its indexing set a proximally non-complete ordered set. Taking the totally bounded uniformity on Y as a restriction of the uniformity on bY , the Samuel compactification sY of such a uniform space coincides with bY . It follows from Proposition 4.1 that $h(x) \in \gamma_p Y$. Since we assumed Y to be proximally complete, we have $\gamma_p Y = Y$ and $h(x) \in Y$. That is true for any $x \in X$, so that $h(X) \subset Y$. Repeating the procedure for h^{-1} instead for h , we get that h maps X onto Y . $\square$

Using the previous general result for known proximally non-complete ordered sets, we get some special results. If a compactification bX of X is given, proximal completeness of X is understood with respect to the proximity on X inherited from bX . If X is a uniform space, we take Samuel compactification sX as its compactification bX .

Corollary 4.3. *Homeomorphisms of compactifications bX, bY of X, Y resp., restrict to homeomorphisms on X onto Y in any of the following cases:*

1. X, Y are proximally complete and have linearly ordered neighborhood bases (in particular, X, Y are Dieudonné complete and $bX = \beta X, bY = \beta Y$).
2. X, Y are complete uniform spaces having linearly ordered uniform bases.
3. Proximally complete X, Y are products of finitely many spaces having linearly ordered neighborhood bases (e.g., products of complete metric spaces).
4. X, Y are products of finitely complete uniform spaces having linearly ordered uniform bases.

The assertion (1) generalizes the classical result by Čech ([3]) for first countable spaces, the assertion (2) the result by Shirota ([16]) for metrizable uniform spaces. Some other applications give generalizations of results by Isbell ([9]) or Mrówka ([12]).

Theorem 4.2 can be used to construct proximally complete ordered sets. Up to now we had just one example, namely free ultrafilters. It is seen from the next result that proximally complete ordered sets are big (or maximal) in some sense. The result is a direct consequence of Proposition 4.1.

Proposition 4.4. *Let X be a uniform space, $\xi \in sX \setminus \gamma_p X$ and $\mathcal{U}_\xi$ be the neighborhood filter of ξ in sX . Then $\mathcal{U}_\xi$ and $\{U \cap X; U \in \mathcal{U}_\xi\}$ or their bases are proximally complete (thus not proximally fine).*

5 Final remarks

In this paper we dealt with products of finitely many ordered sets. The main result then has a direct application to restrictions of homeomorphisms on, e.g., products of finitely many uniform spaces. If we want to have analogous results for, e.g., products of infinitely many uniform spaces we cannot consider products of infinitely many ordered sets but its σ -subproducts. A base of a product of uniform spaces is isomorphic to σ -product of bases of the coordinate spaces. Similarly for topological spaces we must deal with σ -product of local neighborhood filters of points in coordinate spaces. From that reason a right counterpart of Theorem 3.5 is an assertion for σ -products of infinitely many ordered sets. The corresponding result for linearly ordered sets was formulated in [7], Proposition 2.5. Its proof suffers the same problem as mentioned in the preceding section for products of finitely many sets. It is not proved there that the used extensions of proximally continuous maps exist. Although it is possible to show the extensions exist we gave another proof of Theorem 3.5 because it is much simpler. Probably, the same situation holds for σ -products. Unfortunately, a direct modification of the proof of Theorem 3.5 does not work and another procedure must be found.

Recently, topological and uniform spaces being in some sense related to the ordered set ω^ω were investigated by T. Banach (see, e.g., [2]). It would be convenient to know whether the ordered product ω^ω is proximally complete or not. If the answer is in the negative, the applications in Theorem 4.2 could be applied to spaces having, e.g., local bases isomorphic to ω^ω . We conjecture that ω^ω is not proximally complete. It means to show that every proximally continuous map $f : \omega^\omega \rightarrow M$ into a complete metric space M , is

uniformly continuous. A proof of a special case of that situation was sent to me by T. Banach, namely for $M = \mathbb{R}$. His proof uses a Talagrand characterization of meager filters and, probably, cannot be extended to maps into general metric spaces. So the next problem is open.

Problem. *Is the ordered product ω^ω proximally fine?*

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