Chapter 6

1–tilting modules and their applications

In this chapter we restrict ourselves to the particular case of tilting modules of projective dimension 1.

In Section 6.1 we will see that in this case there are substantial simplifications of the general theory, partly thanks to a relatively good general understanding of modules of projective dimension 1, and partly to the fact that all 1–tilting classes are torsion classes.

In Section 6.2 we will provide an explicit description of 1–tilting classes and modules over particular rings. We will start with the case of artin algebras, but our main concern will be the commutative case: we will completely characterize all tilting modules over Prüfer, valuation and Dedekind domains.

In Section 6.3 we will present an application of infinite–dimensional tilting theory to a proof of the following result (Theorem 6.3.16): for any commutative ring R and any multiplicative set S consisting of (some) non–zero divisors, the localization $S^{-1}R$ has projective dimension ≤ 1, if and only if $S^{-1}R/R$ is a direct sum of countably presented modules.

6.1 Tilting torsion classes

0–tilting modules are easily seen to coincide with the projective generators. Now we consider in more detail the 1–tilting modules. Some of their examples have already been presented in Section 5.1.

First we note a number of simplifications:

Lemma 6.1.1. *Let R be a ring, T be a 1–tilting module and $\mathfrak{T} = (\mathcal{A}, \mathcal{B})$ be the cotorsion pair induced by T. Then:*

(a) *$\mathfrak{T}$ is the cotorsion pair generated by T.*

(b) *There is a short exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \operatorname{Add}(T)$ (so we can assume $r = 1$ in condition (T3)).*