1 Introduction and Statement of Main Results

In their 1979 influential paper [32] on layer potential techniques for boundary value problems, E. Fabes, M. Jodeit and N. Rivière have showed that for a bounded domain $\Omega \subset \mathbb{R}^n$ of class $\mathscr{C}^1$, with outward unit normal ν and boundary surface measure σ, the harmonic double layer[1]

$$Kf(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y), \qquad x \in \partial\Omega, \tag{1.0.1}$$

is a compact operator on $L^p(\partial\Omega)$ for each $p \in (1,\infty)$. This continued a long line of work, originating with Erik Ivar Fredholm in the late 1800's and early 1900's whose motivation for the development of Fredholm theory was the use of such compactness results in order to treat boundary value problems via integral equation methods. In particular, this development made it possible to solve in [32] boundary value problems for the Laplacian in a bounded $\mathscr{C}^1$ domain $\Omega \subset \mathbb{R}^n$ equipped with Dirichlet or Neumann boundary conditions. In the formulation of these problems, the boundary data is selected from Lebesgue spaces, the boundary traces are taken in a pointwise nontangential sense, and the size of the solution was measured through the membership of the nontangential maximal function to $L^p(\partial\Omega)$.

Concerning the compactness of K on $L^p(\partial\Omega)$, the demand that $\Omega \subset \mathbb{R}^n$ is of class $\mathscr{C}^1$ may be significantly relaxed and, in [50], the authors have identified the natural class of domains in which principal value singular integral operators with an algebraic structure similar to that of the harmonic double layer K in (1.0.1) induce compact mappings on each $L^p(\partial\Omega)$, $p \in (1,\infty)$. The membership of Ω in the class of domains in question, dubbed REGULAR SKT[2] DOMAINS in [50], is characterized by the following geometric measure theoretic conditions:

$$\begin{aligned}
&\Omega \text{ is an open set with compact Ahlfors regular boundary,}\\
&\Omega \text{ satisfies a two-sided local John condition, and}\\
&\text{the outward unit conormal } \nu \text{ of } \Omega \text{ belongs to } \mathrm{VMO}(\partial\Omega).
\end{aligned} \tag{1.0.2}$$

Here and elsewhere, $\mathrm{VMO}(\partial\Omega)$ stands for the Sarason class of functions of vanishing mean oscillation on $\partial\Omega$, relative to the "surface measure" $\sigma := \mathscr{H}^{n-1}\lfloor\partial\Omega$, where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in the ambient space (viewed as a metric space; cf. [83]). See § 2.2 for definitions of Ahlfors regularity and the John condition. Domains satisfying the conditions listed in (1.0.2) include bounded domains of class $\mathscr{C}^1$ or, more generally, bounded domains locally given as the upper-graphs of continuous functions with gradients in VMO. This being said, regular SKT domains need

1 where $\langle\cdot,\cdot\rangle$ is the standard inner product, and ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$
2 acronym for Semmes-Kenig-Toro