9 Further Tools from Differential Geometry, Harmonic Analysis, Geometric Measure Theory, Functional Analysis, Partial Differential Equations, and Clifford Analysis

Here we discuss some further useful material from differential geometry, geometric measure theory, harmonic analysis, functional analysis, partial differential equations, and Clifford analysis.

Sections 9.1 and 9.2 treat connections on vector bundles over a Riemannian manifold M, first for general vector bundles, then specifically for the bundles TM and $\Lambda^l TM$. In Section 9.3 we introduce the Riemann curvature tensor on such a Riemannian manifold, and an associated Ricci operator, which is seen to arise to compare the Hodge-Laplacian and the Bochner Laplacian.

Sections 9.4 and 9.5 present results on L^p-Sobolev spaces of functions on $\partial\Omega$, first when $\partial\Omega$ is an Ahlfors regular domain in $\mathbb{R}^n$, then when $\partial\Omega$ is an Ahlfors regular domain in a Riemannian manifold M.

Section 9.6 is devoted to various integration by parts formulas involving functions on $\partial\Omega$. Section 9.7 discusses an elliptic regularity result yielding solutions in $H^{1/2,2}(\Omega)$.

Section 9.9 presents basic estimates for single and double layer potentials acting on functions on $\partial\Omega$ when Ω is a UR domain, first for model operators, then for potentials arising from second order elliptic operators with coefficients of limited smoothness. The applicability of such results depends on the invertibility of members of an appropriate class of second order elliptic operators, discussed in Section 9.10.

Our analysis depends not only on boundedness of various operators on $L^p(\partial\Omega)$ from Section 9.9, for UR domains, but also on compactness results for certain important special cases, described in Section 9.11. Here the specialization from UR domains to regular SKT domains comes to the fore. More generally, operators analyzed here are close to compact if Ω is an ε-SKT domain, for small ε.

This chapter ends with three short sections, one on a sharp divergence theorem, one outlining the structure of Clifford algebras, and one on closed, densely defined linear operators T on a Hilbert space, satisfying $T^2 = 0$, discussing the self-adjoint operator $T + T^*$ and the connection with the self-adjoint operator $TT^* + T^*T + I$, of use in the analysis of the Hodge-Laplacian.

9.1 Connections and Covariant Derivatives on Vector Bundles

We aim to describe connections that arise on various vector bundles over a manifold M. We develop this notion for general vector bundles here, in preparation for special