8 Cauchy's problem for 2-D quasi-geostrophic equation

8.1 Introduction

The dissipative quasi-geostrophic equation considered here has the form:

$$\theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = f, \quad x \in \mathbb{R}^2,\ t > 0,$$
$$\theta(0, x) = \theta_0(x), \tag{8.1}$$

where θ represents the potential temperature, $\kappa > 0$ is a diffusivity coefficient, $\alpha \in [\frac{1}{2}, 1]$ is a fractional exponent, and $u = (u_1, u_2)$ is the *velocity field* determined by θ through the relation:

$$u = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right), \quad \text{where } (-\Delta)^{\frac{1}{2}}\psi = -\theta, \tag{8.2}$$

or, in a more explicit way,

$$u = (-R_2\theta, R_1\theta), \tag{8.3}$$

where R_i, $i = 1, 2$ are the *Riesz transforms*.

This chapter is devoted to the global in time solvability and properties of solutions to the Cauchy problem (8.1). A large literature was devoted to that problem through the last 20 years; compare [39, 40, 59, 21, 71, 101, 119, 174, 183, 184] for more references. The basic approach was to obtain a *weak solution* to (8.1) using the *viscosity technique* (e. g., [124]), which means adding the viscosity term $\epsilon\Delta\theta$ to the right-hand side of the equation, solving the regularized problem and letting $\epsilon \to 0^+$. Our approach is different. We consider first a *family of subcritical problems* (8.1) with $\alpha \in (\frac{1}{2}, 1]$, which can be treated in the framework of [32, 86] as semilinear equations with sectorial operator. Thanks to a maximum principle valid for (8.1) (Lemma 8.2.4), we have a *uniform in* $\alpha \in (\frac{1}{2}, 1]$ estimates of solutions to that subcritical problems in $L^p(\mathbb{R}^2)$, $1 \le p \le +\infty$. Letting $\alpha \to \frac{1}{2}^+$ over a sequence of regular $H^{2\alpha^- + s}(\mathbb{R}^2)$, $s > 1$, solutions θ^α to (8.1), this property allows us to introduce in Theorem 8.6.2 a "*weak L^p solution*" of the limiting *critical problem* (8.1). Our considerations relate most closely to the J. Wu papers [183, 184], using however another, semigroup approach of the monographs [32, 86].

8.1.1 Description of the results

The quasi-geostrophic equation (8.1) is a challenging problem to study. Many papers devoted to it were published very recently; some of them are listed in the references; see anyway [101, 102, 117, 188].

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