*Chapter 3*

# Basics of Hyperbolic Groups and Manifolds

## §1. Margulis's Lemma and splittings of hyperbolic manifolds

In the study of hyperbolic groups and manifolds, an essential tool is the Margulis Lemma which describes (roughly speaking) "uniformly thin" parts of hyperbolic manifolds as those submanifolds that have virtually Abelian fundamental groups.

More precisely, for a given Riemannian manifold/orbifold $M$ and $\varepsilon > 0$, we can define the $\varepsilon$-splitting $M = M_{(0,\varepsilon]} \cup M_{[\varepsilon,\infty)}$ of $M$ into $\varepsilon$-thin and $\varepsilon$-thick parts. Here the $\varepsilon$-thin part $M_{(0,\varepsilon]}$ consists of those points $x \in M$ through that there pass homotopically non-trivial (piecewise differentiable) loops $\gamma \subset M$ with length not exceeding $\varepsilon$. By definition, the *$\varepsilon$-thick part* $M_{[\varepsilon,\infty)}$ is the complement in $M$ to the interior $M_{(0,\varepsilon)}$ of the $\varepsilon$-thin part.

Of course, for a compact manifold $M$, we have $M_{(0,\varepsilon]} = \emptyset$ whenever $\varepsilon$ is small enough.

To examine this splitting, it is natural to consider the universal covering $\widetilde{M}$ which, in the case of a complete hyperbolic manifold $M$, is the hyperbolic space $\mathbb{H}^n$, and to consider subgroups $G_\varepsilon(x)$, $x \in \mathbb{H}^n$, of the fundamental group $\pi_1(M) \cong G \subset \operatorname{Isom} \mathbb{H}^n$ generated by all elements $g \in G$ for which the hyperbolic distance $d(g(x), x) \leq \varepsilon$. For a fixed $x$, we define $G'_\varepsilon(x) \subset G$ to be the subgroup of $G_\varepsilon(x)$ generated by the elements whose derivatives also differ by $\varepsilon$ from the identity. Then we can formulate Margulis's Lemma describing the $\varepsilon$-splitting of a hyperbolic manifold/orbifold $M$ as follows:

**Theorem 3.1** (Margulis's Lemma). *For all $n$ there exist $\varepsilon = \varepsilon(n) > 0$ and $\nu = \nu(n) \in \mathbb{N}$ such that, for any discrete group $G \subset \operatorname{Isom} \mathbb{H}^n$ and $x \in \mathbb{H}^n$, the subgroup $G_\varepsilon(x) \subset G$ is virtually Abelian and its subgroup $G'_\varepsilon(x)$ is Abelian of index at most $\nu$.*

Before we go on to prove this statement, we remark that it can be generalized to the case of Riemannian manifolds $X$ having bounded curvature $\kappa$, $-1 \leq \kappa \leq 0$. In this generalization, the property of $G_\varepsilon(x)$ to be a virtually Abelian subgroup of a hyperbolic group $G$ must be replaced by virtually nilpotent, see Ballman–Gromov–