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Divisors on real curves

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Abstract. Let X be a smooth projective curve over $\mathbb{R}$. In the first part, we study effective divisors on X with totally real or totally complex support. We give some numerical conditions for a linear system to contain such a divisor. In the second part, we describe the special linear systems on a real hyperelliptic curve and prove a new Clifford inequality for such curves. Finally, we study the existence of complete linear systems of small degrees and dimension r on a real curve.

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Introduction

In this note, a real algebraic curve X is a smooth proper geometrically integral scheme over $\mathbb{R}$ of dimension 1. A closed point P of X will be called a real point if the residue field at P is $\mathbb{R}$, and a non-real point if the residue field at P is $\mathbb{C}$. The set of real points, $X(\mathbb{R})$, will always be assumed to be non-empty. It decomposes into finitely many connected components, whose number will be denoted by s. By Harnack's theorem we know that $1 \leqslant s \leqslant g+1$, where g is the genus of X. A curve with $g+1-k$ real connected components is called an $(M-k)$-curve.

The group $\mathrm{Div}(X)$ of divisors on X is the free abelian group generated by the closed points of X. Let $D \in \mathrm{Div}(X)$ be an effective divisor. We may write $D = D_r + D_c$, in a unique way, such that D_r and D_c are effective and with real, respectively non-real, support. We call D_r (resp. D_c) the real (resp. non-real) part of D. In the sequel, we will say that D is totally real (resp. non-real), if $D = D_r$ (resp. $D = D_c$).

By $\mathbb{R}(X)$, we denote the function field of X. Let $\mathrm{Pic}(X)$ denote the Picard group of X, which is the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors, i.e. divisors of elements in $\mathbb{R}(X)$. Since a principal divisor has an even degree on each connected component of $X(\mathbb{R})$ ([4] Lemma 4.1), we may introduce the following invariants of X:

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