Universal Weil Divisors

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1. Introduction

It is well known that on a smooth scheme, a positive divisor is the same thing as a subscheme defined by an invertible ideal. The parametrization problem — construction of a universal family — for families of divisors on a smooth projective scheme has been solved by Grothendieck's construction of Hilbert schemes, [3]. On the other hand, Matsusaka [6] has proved the existence of maximal families of positive divisors on a projective variety. His construction, however, does not arise from representing a functor.

Our purpose here is to represent a certain functor $\mathscr{W}$ of families of 'Weil divisors' on suitable projective schemes. Our notion of 'relative Weil divisor', or family of Weil divisors, is analogous to that of relative Cartier divisor discussed in [7], and reduces to the latter in the case where the carrier scheme is smooth. In the case where the carrier is nonsingular in codimension $\leqq 1$ — condition R_1 of [2] — we represent the functor $\mathscr{W}$ on the category of *normal* noetherian schemes (over a base field k). If the carrier satisfies conditions R_2 and S_2 of [2], then we can represent $\mathscr{W}$ on the category of *reduced* noetherian schemes over k. In either case the universal parameter schemes are disjoint unions of projective schemes — one for each value of the degree of the divisors considered.

Presumably, the situation in the R_1 case cannot be improved, at least by our methods, but it appears to be an open question as to whether conditions R_2 and S_2 are sufficient to represent $\mathscr{W}$ on the category of all noetherian k-schemes. It will also be evident to the schematically adept reader that our assumption that the base is a field could be dropped at the cost of lengthier statements and more verbose proofs.

2. Divisors and Cartier divisors

Let X be a noetherian scheme. Let $X_{(n)}$ denote the set of $x \in X$ such that $\mathcal{O}_{X,x}$ has Krull dimension n, and X_n the set of $x \in X_{(n)}$ such that $\mathcal{O}_{X,x}$ is regular. Let reg $X = \bigcup_n X_n$. For each $x \in X$, and each open $U < X$, let $\mathbf{Z}_x(U) = \mathbf{Z}$ if $x \in U$, $\mathbf{Z}_x(U) = (0)$ if $x \notin U$. We set

$$\mathscr{C}_{n,X} = \bigoplus_{x \in X_{(n)}} \mathbf{Z}_x,$$

$$\mathscr{Z}_{n,X} = \bigoplus_{x \in X_n} \mathbf{Z}_x.$$