On the quadratic character of some quadratic surds

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In this paper we shall consider pairs of primes $p_1 \equiv p_2 \equiv 1 \pmod 4$ which are quadratic residues of each other so that $\sqrt{p_1}$ modulo p_2 and $\sqrt{p_2}$ modulo p_1 are rational integers. In other words we let

$$p_1 = a_1^2 + 4b_1^2 \text{ and } p_2 = a_2^2 + 4b_2^2 \text{ with } \left(\frac{p_1}{p_2}\right) = 1. \tag{1}$$

We shall be concerned with the quadratic character of the fundamental units

$$\varepsilon_1 = \frac{(t_1 + u_1\sqrt{p_1})}{2} \text{ and } \varepsilon_2 = \frac{(t_2 + u_2\sqrt{p_2})}{2} \tag{2}$$

which correspond to the least positive solutions of the Pell equation

$$t^2 - pu^2 = -4$$

for $p = p_1$ and p_2 respectively.

We shall show that these characters are the same as those of

$$\alpha_1 = \frac{(a_1 + \sqrt{p_1})}{2} \text{ and } \alpha_2 = \frac{(a_2 + \sqrt{p_2})}{2} \tag{3}$$

as well as those of some other related algebraic integers. We choose the signs on the square roots so as to avoid zero characters.

In a previous paper [7] we discussed the special case $p_2 = 5$ and gave a very elementary proof that the Fibonacci unit $\frac{(1+\sqrt{5})}{2}$ is a quadratic residue modulo p_1 if and only if:

$$\begin{aligned} &\text{either 5 is a quartic residue of } p_1 = 20n + 1 \\ &\text{or 5 is a quartic non-residue of } p_1 = 20n + 9. \end{aligned} \tag{4}$$

This statement can be written in the form of a quartic reciprocity law and is in fact a special case of a statement found in Scholz [8], namely

„ε_1 ist dann und nur dann quadratischer Rest für p_2, wenn p_1 und p_2 gegenseitige biquadratische Reste oder Nichtreste sind."

Since this is part 4 of a 5-part theorem involving class-field theory, which apparently remained unnoticed for some 35 years, it seems worth while to give an elementary proof based on cyclotomy.