Extension of topological homology theories to partially ordered sets

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Introduction

In this paper I define homology and cohomology theories of partially ordered sets (posets). The singular, simplicial, and Čech-homology and cohomology theory of topological spaces is generalized to posets. In a categorical sense, which is made precise later, each of these *homology theories of posets* induces the corresponding theory of topological spaces; the following example may illustrate the situation: I associate with a topological space X the set T_X of open subsets of X, partially ordered by the set inclusion. Then the q-th singular homology group (integer coefficients) of the pair of topological spaces (X, A) is isomorphic to what I call the q-th *singular homology group* of the *pair of posets* (T_X, T_A), (I, § 2, 3. 1, 3. 2, § 9). This shows in particular that the singular homology groups on a topological space X are completely determined by the poset T_X.

In the first Chapter I define the *singular homology groups* of a poset (§ 2) using an object which I construct, the filter set of a poset (§ 1). I also use the filter set to define, what I call a homomorphism of posets (I, 1. 3). I show that the category $\mathfrak{C}$ of posets and homomorphisms form a homology category in the sense of [2], IV, 9. 1, p. 117. Then by a standard procedure ([2], V, § 11, 12, p. 150—156) we have homology- and cohomology groups of a *pair of posets* with coefficients in an arbitrary module over an arbitrary ring with unit element. The universal coefficient theorem for an arbitrary module over an arbitrary ring with unit element applies, as it was pointed out to me by S. Lubkin, since the *singular chain complex* of a poset (I, § 2) is free. Considering the category of topological spaces and continuous maps $\mathfrak{D}$ the following holds: there exists a homology preserving functor ([2], IV, 9. 4, p. 118) from $\mathfrak{D}$ into $\mathfrak{C}$ (I, § 9).

The concept of a *covering* introduced in I, 1. 7 is used in the second chapter in two directions. Since the set of coverings on a poset form a directed set (II, 5. 2) the *Čech-homology* and *cohomology groups* can be defined (II, § 5). These groups may directly be related to the groups of [4]. The coverings itself are used to define *simplicial posets*, *simplicial homomorphisms* and the *simplicial homology*, (II, § 1, 1. 1, 1. 4). As in the first Chapter I construct in the second chapter for the category of simplicial complexes (with condition (A), II, § 1) and simplicial maps a homology preserving functor into the category of simplicial posets and simplicial homomorphisms. A relation between the homology and cohomology groups on categories in [5] and my simplicial homology and cohomology groups is given by the *subdivision of a simplicial structure* on a poset, (II, § 4).