On a class of multiplicative arithmetic functions

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1. Introduction

If $f(n)$ is a complex-valued function of the real variable n and $f(n) = 0$ when n is not a positive integer, then $f(n)$ is called an arithmetic function. An arithmetic function $f(n)$ is said to be multiplicative in n if

$$(1.1) \qquad f(n)f(m) = f(nm)$$

whenever $(n, m) = 1$. It follows that a multiplicative function $f(n)$ (or briefly f) is determined if the values of $f(p^e)$, p a prime and e an integer $\geqq 1$, are known. Further, if $f(n) \neq 0$ identically, then $f(1) = 1$. If the relation (1. 1) holds for all pairs of values of n, m, then f is referred to as a *completely multiplicative function.* For a completely multiplicative function f and for a prime p, $f(p^e) = \{f(p)\}^e$.

In this note, we confine ourselves to the class of multiplicative functions f which satisfy

$$(1.2) \qquad f(n)f(m) = \sum_{d|(n,m)} f\left(\frac{nm}{d^2}\right) g(d)$$

where the summation extends over all the common divisors d of n, m, and g is completely multiplicative. Such functions may be termed as *specially multiplicative* functions [3]. An equivalent form of (1. 2) is

$$(1.3) \qquad f(nm) = \sum_{d|(n,m)} f(n/d)f(m/d)g(d)\mu(d)$$

where $\mu(n)$ is the Möbius Function. The most general solution of (1. 2) may be found by assigning arbitrary values to $f(p)$ and determining recursively by

$$(1.4) \qquad f(p^{e+1}) = f(p)f(p^e) - g(p)f(p^{e-1}); \qquad e \geqq 1.$$

It may be mentioned that the divisor function $d(n)$ and the sum function $\sigma(n)$ of the divisors of n belong to the class of specially multiplicative functions.

The *composition* [5] of two multiplicative functions f, h is defined by

$$(1.5) \qquad (f \cdot h)(n) = \sum_{d|n} f(d)h(n/d)$$

where the summation is over all divisors d of n. Vaidyanathaswamy [5] has shown that a multiplicative function f admits an identity of the type (1. 2) if and only if it is the composite of two completely multiplicative functions.