Geodesics on the tangent sphere bundles over space forms

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Introduction

By a space form we mean in this paper any one of the Euclidean n-space E^n, the unit n-sphere $S^n[1]$ in E^{n+1} and the hyperbolic n-space $H^n[-1]$ of sectional curvature -1. For brevity, we denote these manifolds by E^n, S^n and H^n respectively. If we denote the set of unit tangent vectors of a space form M^n by $T_1(M^n)$, then $T_1(M^n)$ with natural topology is the total space of the tangent sphere bundle $\pi: T_1(M^n) \to M^n$. The set $T_1(M^n)$ keeps a natural Riemannian metric induced from the original metric of M^n [3]. Each geodesic Γ of $T_1(M^n)$ is then interpreted as a certain vector field y along a curve $C = \pi\Gamma$ in M^n. The purpose of this paper is to characterize Γ in terms of these y and C. The special case of $T_1(S^2)$ is found in [2].

We shall give a short summary of our results. Geodesics on $T_1(M^n)$ can be classified into horizontal, vertical and oblique types. Any geodesic of horizontal type is a parallel vector field along a geodesic on M^n (Theorem 1). Any geodesic of vertical type is a great circle on a fibre (Theorem 2).

Geodesics of oblique type can be classified into three classes.

(i) The class of geodesics of oblique type over geodesics in M^n. This class of geodesics may exist only for $n \geqq 3$. Each geodesic of this class is represented by a unit vector field which moves helicoidally along a geodesic in M^n (Theorems 3, 5 and 12 for $T_1(E^n)$, $T_1(S^n)$ and $T_1(H^n)$ respectively). For $T_1(E^n)$ only this class of geodesics can occur. Some geodesics of this class in $T_1(S^n)$ may be closed geodesics (Theorem 6).

(ii) The class of geodesics over curves of constant first curvature $\varkappa_1 (>0)$ and vanishing second curvature ($\varkappa_2 = 0$) in M^n. In S^n the latter curves are small circles and in H^n, they are equidistant curves, horocycles and proper circles according as $\varkappa_1 < 1$, $\varkappa_1 = 1$ and $\varkappa_1 > 1$ respectively. For $T_1(S^2)$ any geodesic of this class is represented by a unit vector field along a small circle which makes constant angle with the circle and is a closed geodesic ([2] Theorem 2). For $T_1(H^2)$ it is represented by a vector field along an equidistant curve, a horocycle or a proper circle which moves by a certain rule (Theorem 9). A criterion for closedness of this geodesic is given in Theorem 10. A characterization of a vector field along any one of curves with constant $\varkappa_1 (>0)$ and $\varkappa_2 = 0$ in S^n or H^n ($n \geqq 3$) which represents a geodesic on $T_1(S^n)$ or $T_1(H^n)$ is given in Theorems 7 and 13 for