Kulikov's criterion for modules

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1. Introduction

The well-known Kulikov criterion (see [6], Theorem 11. 1) characterizes abelian p-groups that are direct sums of cyclic groups. Prüfer Theorems ([6], Theorems 11. 2, 11. 3) follow from the Kulikov criterion as simple corollaries. T. S. Shores [11] calls a cyclic right R-module mR ideal-cyclic if the right annihilator of m in R is a two-sided ideal. He also characterized the class of (non-commutative) rings with the property that some versions of the first Prüfer Theorem hold for right R-modules. The notion of ideal-cyclic module seems to be useful for the further investigations. Our main purpose is to characterize the class of (non-commutative) rings having the property that a version of Kulikov criterion holds for right R-modules.

Moreover, we shall show that the Kulikov criterion holds for primary right R-modules if and only if the second Prüfer Theorem is valid for primary right R-modules. We shall also derive some further properties of direct sums of ideal-cyclic modules and basic properties of quasicyclic modules.

2. Preliminaries

In what follows R denotes an associative ring with identity. Unless otherwise stated, by the word module we shall always mean a unitary right R-module. A module M is said to be simple if it is non-zero and has no proper submodules. The socle $S(M)$ of the module M is the submodule of M generated by all the simple submodules of M. The Loewy series of M (often called socle sequence in the literature) is defined for the ordinals by

$$S_1(M)=S(M),\ S_{\alpha+1}(M)/S_\alpha(M)=S(M/S_\alpha(M)) \quad \text{and} \quad S_\alpha(M)=\bigcup_{\beta<\alpha} S_\beta(M),$$

where α is a limit ordinal. The least ordinal $\alpha=\alpha(M)$ for which $S_\alpha(M)=S_{\alpha+1}(M)$ is the Loewy length of M. M is a Loewy module if $S_\alpha(M)=M$ for some α. A Loewy module with finite Loewy length is said to be bounded.

Let I be a maximal right ideal of R. The I-socle $S(M, I)$ of a module M is the submodule of M generated by all the simple submodules of M isomorphic to R/I. As before, one can define the I-Loewy series, I-Loewy module, I-Loewy length.