On a problem of Narkiewicz

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§ 0. Introduction

In [1], W. Narkiewicz considered the following problem: given a quadratic number field K, it is required to obtain an asymptotic formula for the number of positive rational integers $\leqq x$ which have unique factorisation into irreducible integers in K. Narkiewicz obtained the required formula, and in [2] generalised the formula to cover the case where K is cyclic of prime degree over the rational field $\mathbb{Q}$. On the basis of these results, one might conjecture that the number of positive rational integers $\leqq x$ with unique factorisation in a normal extension K is asymptotically

$$\frac{A(K)\,x}{(\log x)^{B(K)}}(\log\log x)^{C(K)}. \tag{0. 1}$$

Here the implied constants depend only on K. Indeed, in [1] Narkiewicz obtained a O-result of the same functional form as (0. 1). In this paper, we show that (0. 1) is subtantially correct, and moreover that the exponents B and C cannot in general depend only on n and h, the degree and class number of K. In fact, we can also dispense with the hypothesis of normality of K over $\mathbb{Q}$; thus, in § 4, we prove the general

Theorem 1. *Let K be any finite extension of the rational field $\mathbb{Q}$; then the number of positive rational integers $\leqq x$ which have unique factorisation in K is asymptotically*

$$\frac{A(K)\,x\,(\log\log x)^{C(K)}}{(\log x)^{B(K)}}\left\{1+O_K\left(\frac{1}{\log\log x}\right)\right\}, \tag{I}$$

where the implied constants depend only on K.

We can be more precise about the constants $B(K)$ and $C(K)$; in § 4, we show that $C(K)<h$, where h is the class number of K, while $B(K)$ is a non-negative rational number <1, which is 0 if and only if $h=1$; it is related in an essential way to a family of conjugacy classes of the Galois group $\mathrm{Gal}\left(H(\bar{K})/\mathbb{Q}\right)$, where $H(\bar{K})$ is the Hilbert class field of $\bar{K}$, the normal hull of $K/\mathbb{Q}$ (see § 2).

Finally, in § 5, we show that in the normal case, the constants $B(K)$ and $C(K)$ cannot depend only on h and $n=[K:\mathbb{Q}]$.