On the existence of meromorphic solutions of differential equations having arbitrarily rapid growth

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1. Introduction

In this paper, we are concerned with the growth of meromorphic solutions of first-order algebraic differential equations, i.e. equations of the form

$$\Omega(z, y, y') = \sum f_{kj}(z)\, y^k (y')^j = 0, \tag{1}$$

where $\Omega(z, y, y')$ is a polynomial in y and y', whose coefficients $f_{kj}(z)$ are meromorphic functions in a region of the plane.

In [13] and [14], G. Valiron proved that when all the coefficients $f_{kj}(z)$ are polynomials, then any entire solution or any solution analytic in the unit disk of equation (1), must be of finite order of growth. These results were extended to meromorphic solutions by A. A. Gol'dberg in [5]. In [5], Theorem 1, Gol'dberg showed that in the case of polynomial coefficients, any meromorphic solution in the plane of equation (1) must be of finite order of growth. In Theorem 3 (and the remark following it) of [5], Gol'dberg showed that for any first-order algebraic differential equation of the form $\sum_{j=0}^{m} Q_{m-j}(z,y)\,(y')^j = 0$, where $Q_0(z,y)$ is a polynomial in z and y, while for $k > 0, Q_k(z,y)$ is a polynomial in y whose coefficients are analytic functions in the unit disk of H^p-class (see [12], p. 314), then any meromorphic solution in the unit disk is of finite order of growth.

The latter result of Gol'dberg raises a natural question, namely, for the class of equations (1) whose coefficients are bounded analytic functions in the unit disk, does any uniform growth estimate hold for all meromorphic solutions in the disk of such equations? In this paper (Theorem 1 below), we answer this question in the negative by showing that equations of the form (1), whose coefficients are bounded analytic functions in the disk, can possess meromorphic solutions of arbitrarily rapid growth in the disk. More precisely, we show that given *any* increasing function $\Psi(r)$ on $[0, 1)$, there exists a meromorphic function $h(z)$ in the unit disk, such that $h'(z)/h(z)$ is of bounded characteristic in the disk and such that for some $r_0 < 1$, $T(r, h) > \Psi(r)$ for all r satisfying $r_0 < r < 1$.