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Localization of tight closure and modules of finite phantom projective dimension

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1. Introduction

In this paper we prove that tight closure commutes with localization in a number of important special cases. By applying Theorem (7.18) of [HH 9] we then obtain at once that, in these same cases, tight closure commutes with smooth base change quite generally.

Throughout this paper, unless otherwise specified, rings are assumed to be commutative Noetherian rings with identity and modules are assumed to be finitely generated and unital. For the most part, we shall be working in positive prime characteristic p.

The notion of *tight closure* for an ideal or submodule, both for Noetherian rings of positive prime characteristic p and for finitely generated algebras over a field of characteristic 0, was introduced in [HH 4]. Expository accounts are given in [HH 1]–[HH 3] and [Hu 1]. The theory is further explored in [HH 6], [HH 8], [HH 9], and [HH 10], as well as in [Ab]. In particular, [HH 8] contains a detailed study of the notion of *phantom homology* (homology is *phantom* when the cycles are in the tight closure of the boundaries in the module of chains), which leads to the theory of modules of *finite phantom projective dimension* developed in [Ab], an important tool here. The definition of finite phantom projective dimension and the parts of the theory that we need are reviewed briefly in § 5.

We shall make strong use of the results of [HH 9], which deals both with the existence of test elements (see (2.5), (2.6) and (2.7)) and with the behavior of tight closure under smooth base change.

Tight closure methods have yielded new proofs (with new insights) that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay (cf. [HR 1], [HR 2], [Ke], [B], [HoE] and [HH 4]), of the Briançon-Skoda theorem on inte-

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