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Strong shift equivalence and K_2 of the dual numbers

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Abstract. This paper discusses the shift equivalence vs. strong shift equivalence problem for primitive subshifts of finite type using an invariant in the algebraic K-theory group $K_2(\Lambda[t]/(t^{m+1}))$ for $\Lambda = \mathbb{Z}$ and $m = 1$.

1. Introduction

Strong shift equivalence theory grew out of R. F. Williams' work [Wi1] on classification of subshifts of finite type, which appear in a wide range of subjects from dynamical systems, to statistical mechanics, to C^*-algebras, to coding theory, and more recently to topological quantum field theory. Good general background references are [K], [LM], [R]. A sample of articles is [A], [BH2], [CK2], [E], [G], [M], [MH], [Sm].

The *full n-shift* is the set X_n of bi-infinite sequences $x = \{x_k\}$ where each x_k comes from a set S of symbols of cardinality n. Typically, $S = \{0, 1, \ldots, n-1\}$. It is equipped with the product topology making it a Cantor set. The shift homeomorphism $\sigma_n : X_n \to X_n$ is defined by $\sigma_n(x)_k = x_{k+1}$. An $n \times n$ zero-one matrix A determines a *subshift of finite type* (X_A, σ_A) of (X_n, σ_n) by letting X_A be the subspace of sequences satisfying $A(x_k, x_{k+1}) = 1$ for all $-\infty < k < \infty$. A is the *transition matrix*. We let $\sigma_A = \sigma_n | X_A$. More generally, the standard edge path construction [LM] allows one to obtain a subshift of finite type (X_A, σ_A) from any $m \times m$ nonnegative integral matrix A. Suppose $f : X \to X$ is a homeomorphism (i.e., a reversible discrete dynamical system), and suppose we have written X as a disjoint union of sets $X = U_1 \cup U_2 \cup \cdots \cup U_n$. Define $\pi : X \to X_n$ as follows: $\pi(x)_k = i$ iff $f^k(x)$ is in U_i. Then $\pi f = \sigma_n \pi$. If $\{U_1, \ldots, U_n\}$ happens to be a Markov partition, then π is a topological conjugacy between (X, f) and the subshift of finite type (X_A, σ_A) where $A(i, j) = 1$ iff $U_i \cap f^{-1} U_j$ is not empty. If one Markov partition exists for (X, f), then there are infinitely many. So we have the fundamental and still open

Classification Problem. *Given nonnegative integral matrices A and B, when are the corresponding subshifts of finite type (X_A, σ_A) and (X_B, σ_B) topologically conjugate?*

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