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On tame towers over finite fields

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Abstract. We discuss the asymptotic behaviour of the genus and the number of rational places in towers of function fields over a finite field.

1. Introduction

The theory of equations over finite fields is a basic topic in classical number theory. Its foundations were laid (among others) by Fermat, Euler, Lagrange, Gauss and Galois. The object of the first investigations in this theory were congruences of the special form

$$y^2 \equiv f(x) \quad \text{(modulo a prime number)},$$

where $f(x)$ is a rational function with integer coefficients. Assuming an analogue of Riemann's hypothesis for the zeta function that he introduced, E. Artin conjectured an upper bound for the number of solutions for such congruences. The general solution of that conjecture was given by A. Weil (the elliptic case being settled before by H. Hasse), and it can be stated as follows: Let F be a function field over the finite field $\mathbb{F}_q$ with q elements, let $N(F)$ denote its number of $\mathbb{F}_q$-rational places and $g(F)$ denote its genus. Then the celebrated theorem of A. Weil [23] states that the following inequality holds:

$$N(F) \leqq q + 1 + 2g(F) \cdot \sqrt{q}.$$

Ihara [13] noticed that one has a strict inequality above if $g(F) > \sqrt{q}(\sqrt{q} - 1)/2$. Setting

$$N_q(g) = \max\{N(F) \mid F \text{ is a function field over } \mathbb{F}_q \text{ with } g(F) = g\}$$

and

$$A(q) = \limsup_{g \to \infty} N_q(g)/g,$$

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