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Banach-Lie quotients, enlargibility, and universal complexifications

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Abstract. We characterize those real Banach-Lie groups which admit universal complexifications, and present examples of Banach-Lie groups which have none. To achieve these goals, we prove new results concerning the enlargibility of Banach-Lie algebras, and derive a necessary and sufficient condition for the existence of Lie group structures on quotients of Banach-Lie groups.

Introduction

In this article, we address several interrelated problems in the theory of Banach-Lie groups, namely: (a) the existence of Lie group structures on quotient groups; (b) enlargibility of Banach-Lie algebras; (c) the existence of universal complexifications of Banach-Lie groups.

A classical fact in the theory of Banach-Lie groups asserts that the topological quotient group G/N of a real Banach-Lie group G by a normal Lie subgroup N can be made a real Banach-Lie group if N is a *split* Lie subgroup, i.e., provided $\mathbf{L}(N)$ is complemented in $\mathbf{L}(G)$ as a topological vector space ([Ms], [Bo], [Up], [DG]; see Section 1 below for the terminology). As our first main result, we show that the assumption that N be split is superfluous (Corollary II.4):

1. Quotient Theorem. *If G is a real Banach-Lie group and N a closed normal subgroup of G, then the topological quotient group G/N can be given a real Banach-Lie group structure if and only if N is a Lie subgroup of G.*

Equipped with the Quotient Theorem, we turn to enlargibility questions of Banach-Lie algebras. Since the fundamental work of van Est and Korthagen [EK], it is known that there are Banach-Lie algebras which are not enlargible, i.e., which are not the Lie algebra of any Banach-Lie group. Van Est and Korthagen also proved the following Enlargibility Criterion: *a Banach-Lie algebra $\mathfrak{g}$ is enlargible if and only if its period group $\Pi(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$ is*

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