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Counter-examples to the Kotzig problem*

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Abstract — As is shown in [1] any bipartite multigraph G has the following property. If for some regular edge-colouring of G with n colours, $n > 3$, all the subgraphs generated by the edges of any three colours are uniquely 3-colourable, then G is uniquely n-colourable.

In the present paper we show that, for any $n \geq 4$ and $m \geq 3$, there exist n-regular multigraphs with $2m$ vertices which do not possess the mentioned property. These multigraphs are counter-examples to the Kotzig problem [2] on transformations of edge-colourings of regular multigraphs.

We consider finite non-oriented multigraphs without loops. (Harary's terminology [3] is used.) Let G be an n-regular multigraph with $2m$ vertices, $n > 3$. A subset of m pairwise non-adjacent edges of G is called a 1-factor of G. A colouring in which each edge is coloured in one of n colours and adjacent edges are coloured in different colours is called a regular colouring of edges of G with n colours or simply n-colouring. For n-colouring of G, a subgraph, which is generated by all the edges of some k colours, is called a k-colour subgraph. A multigraph G is called n-colourable if it has at least one n-colouring.

In 1975 Kotzig [2] set the following problem. Let G be an n-regular n-colourable multigraph, $n > 3$. Can all the n-colourings of G be obtained from any n-colouring of G by transformations of two-coloured and three-coloured subgraphs such that all the intermediate colourings are n-colourings? As is shown in [4, 5] the answer is positive if G is a bipartite multigraph.

In the present paper we show that in the general case the answer to the Kotzig problem is negative.

Two n-colourings of G are called isomorphic if one can be obtained from the other by renaming the colours. A multigraph G is called uniquely n-colourable if all its n-colourings are isomorphic to each other under the condition that the parallel edges joining a pair of vertices are assumed to be equal. (In [6] uniquely n-colourable multigraphs are investigated under the condition that parallel edges are assumed to be different.) In [1] it is shown that any bipartite multigraph has the following property. If for some n-colouring of G, $n > 3$, all the three-coloured subgraphs are uniquely 3-colourable, then G is uniquely n-colourable.

Let $\Phi(2m, n)$ denote the set of all the n-regular multigraphs having $2m$ vertices and possessing the following two properties.

Property 1. A multigraph G has at least two non-isomorphic n-colourings.

Property 2. A multigraph G has an n-colouring such that all the three-coloured subgraphs of G are uniquely 3-colourable. It is clear that then all the two-coloured subgraphs of G are uniquely 2-colourable.

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