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On transportation polytopes with the minimum number of k-faces*

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Abstract — Criteria of belonging of a non-degenerate transportation polytope with a given number of faces (of maximum dimension) to the class of polytopes with the minimum number of k-faces of all dimensions (beginning with zero) are suggested. A formula for this number is obtained.

1. INTRODUCTION

One of the basic problems of combinatorial theory of polyhedra (going back to Euler) deals with the description of the range of values of the vector function $f(M) = (f_0(M), f_1(M), \dots, f_{d-1}(M))$ whose kth component is equal to the number of k-faces of a d-polyhedron M. Up to now this problem is solved only for the classes of d-polyhedra with the number of vertices not greater than $d+3$, and also for polytopes of different combinatorial types: simplexes, prisms, pyramids [1]. The problem of estimating the bounds for the variation of components of the vector $f(M)$, provided that the other components are fixed, is also known (see [1, 2]). Most often the number of faces (of maximum dimension) is fixed, and one tries to obtain bounds for the other components. Such investigations are carried out both for abstract polyhedra and for the polyhedra of some combinatorial optimization problems.

A criterion of belonging of the transportation polytope (TP)

$$M(a,b) = \Big\{ x = \|x_{ij}\|_{m\times n}:\ x_{ij} \geq 0 \quad \text{for all} \quad (i,j) \in \mathbf{N}_m \times \mathbf{N}_n,$$
$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j \in \mathbf{N}_n, \quad \sum_{j=1}^{n} x_{ij} = a_i \quad i \in \mathbf{N}_m \Big\}$$

of order $m \times n$, defined by vectors $a = (a_1, a_2, \dots, a_m)$, $b = (b_1, b_2, \dots, b_n)$ with real positive components ($m, n > 1$, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$), to the class of non-degenerate TP's with a given number of faces and the maximum number of k-faces ($k = 0, 1, \dots, d-2$) is given in [3, 4]. Here and in the sequel $\mathbf{N}_t = \{1, 2, \dots, t\}$, $d = \dim M(a,b) = (m-1)(n-1)$. An explicit formula for the maximum number of k-faces is still unknown. Only the formulae for some special cases are known (see, for example, [1, 5, 6]).

In this paper criteria for belonging of a non-degenerate TP of order $m \times n$ with given number of faces to the class of polytopes with the minimum number of k-faces, $k = 0, 1, \dots, d-2$, are suggested and a formula for this number is obtained. These results generalize all the theorems which were obtained previously for special cases. The starting point for these results is the inclusion–exclusion formula [7] together with the notion of spectrum of two TP's, which was first introduced in [8].

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