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On the number of summands in the Hilbert–Kamke problem in prime numbers*

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Abstract — In this paper it is proved that the number of summands s, which are required for the simultaneous representations of positive integers N_j, $1 \le j \le n$, satisfying the corresponding necessary arithmetic conditions, as sums of the jth powers of prime numbers $x_i > n + 1$, $1 \le i \le s$, belongs to some residue class modulo $R_0(n) = \exp\{n \ln n + O(n)\}$, moreover, if $n \ge 17$, then for every class of numbers $N_1, \dots, N_n$, corresponding to s modulo $R_0(n)$, the least s, which is sufficient for these representations, is determined from the inequalities $s_0(n) \le s \le s_0(n) + R_0(n) - 1$, where $s_0(n) \sim 3^{\alpha_n}$, $\alpha_n \sim 3n/4$, $n \to \infty$, provided that the numbers $N_1, \dots, N_n$ satisfy some order conditions and are large enough.

The analogous situation has arisen for simultaneous representations of $N_1, \dots, N_n$ as sums of powers of arbitrary prime numbers.

1. INTRODUCTION

Let $n \ge 2$, s, $N_1, \dots, N_n$ be positive integers, and let J be the number of solutions of the system of equations

$$x_i^j + \dots + x_s^j = N_j, \qquad j = 1, \dots, n, \tag{1}$$

in prime numbers, and $P = N_n^{1/n}$. For fixed $n \ge 11$, $s \ge 2n^2(3 \ln n + \ln \ln n + 4) - 21$ the formula

$$J = \gamma \sigma_1 P^{s-n(n+1)/2} (\ln P)^{-s} + O(P^{s-n(n+1)/2} \ln \ln P (\ln P)^{-s-1})$$

was established in [1], here $\gamma = \gamma(N_1,\dots,N_n)$ is a singular integral and $\sigma_1 = \sigma_1(N_1,\dots,N_n)$ is a singular series, $0 \le \gamma \ll 1$, $0 \le \sigma_1 \ll 1$.

The singular integral γ converges absolutely for $s > n(n + 1)/2 + 1$ and the same restriction on s provides the fulfilment of the relation $\gamma \gg 1$ under some sufficiently general order conditions on $N_1, \dots, N_n$ (see [2]). By analogy with the singular series of the asymptotic formula for the number of solutions of system (1) in positive integers (see [3]) the series σ_1 converges absolutely for $s > n(n + 1)/2 + 2$.

Arithmetical conditions of different form on $N_1, \dots, N_n$, s, which are necessary for the positiveness of σ_1, was found by Chubarikov [4] and the author [5]. Note that those conditions are sufficient for fulfilment of the relation $\sigma_1 \gg 1$, if s exceeds some lowest bound $s_0(n)$. Note also that those conditions are necessary for solvability of system (1) in prime numbers greater than $n + 1$, as well as sufficient, if the conditions $s \ge s_0(n)$, $\gamma \gg 1$ are fulfilled and P is large enough.

The upper bound $s_0(n) \ll n^3 2^{2n}$ for the value $s_0(n)$ was given in [4] (with absolute constant into the symbol $\ll$). The formula for $s_0(n)$ announced by the author in [5] is erroneous, since it corresponds only to the lowest bound for positiveness of the 2-adic

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