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Combinatorial approach to enumeration of doubly stochastic non-negative integer square matrices*

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Abstract — Let $H_R(n,r)$ be equal to the number of $n \times n$ matrices with non-negative integer elements such that all row sums and all column sums are equal to r and all elements with indices from a set R are equal to zero. We investigate the properties of the function $H_R(n,r)$ and give a combinatorial interpretation of the obtained results.

1. INTRODUCTION

In [1] Kenji Mano investigated the number $H(n,r)$ of different ways to allocate nr objects of n types with r objects of each type among n persons such that each person receives r objects. The number $H(n,r)$ may be interpreted as the number of $n \times n$ matrices (a_{ij}) with non-negative integer elements which satisfy the conditions

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = r. \tag{1.1}$$

In all subsequent papers $H(n,r)$ is the number of such matrices.

In [2] the following hypothesis (ADG hypothesis) was proposed: for any n and r

$$H(n,r) = \sum_{i=0}^{N} c_i \binom{r+n+i-1}{n+2i-1}, \tag{1.2}$$

where $N = \binom{n-1}{2}$ and c_i depend on n and i only.

Representation (1.2) was proved by Stanley [3, 4] and Ehrhart [6].

The literature on the ADG hypothesis and its generalizations is quite extensive, we note only the papers [1–12, 14–18]. In [12] a combinatorial approach to evaluation of $H(n,r)$ was suggested.

Let R be a fixed set of entries of an $n \times n$ matrix and let $H_R(n,r)$ be the number of $n \times n$ matrices (a_{ij}) with non-negative integer elements satisfying conditions (1.1) and such that $a_{ij} = 0$ for all $(i,j) \in R$.

As it follows from Stanley's papers [3, 4], the function $H_R(n,r)$ is a polynomial in r for fixed R and n.

In this paper we suggest a combinatorial approach to evaluation of the polynomials $H_R(n,r)$, which generalizes the approach from [12], and give combinatorial interpretations of the degree of the polynomial $H_R(n,r)$ and of the coefficients of the expansion

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