

Uniscalar p -adic Lie groups

Helge Glöckner and George A. Willis¹

(Communicated by Dan Segal)

Abstract. A totally disconnected, locally compact group G is said to be *uniscalar* if its scale function $s_G : G \rightarrow \mathbb{N}$, as defined in [G. A. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. **300** (1994), 341–363], is identically 1. It is known that G is uniscalar if and only if every element of G normalizes some open, compact subgroup of G . We show that every identity neighbourhood of a compactly generated, uniscalar p -adic Lie group contains an open, compact, normal subgroup. In contrast, uniscalar p -adic Lie groups which are not compactly generated need not possess open, compact, normal subgroups.

1991 Mathematics Subject Classification: 22E20; 20E08, 20F50.

1 Introduction

Following Palmer [8], we say that a totally disconnected, locally compact group G is *uniscalar* if its scale function $s_G : G \rightarrow \mathbb{N}$ is identically 1, or, equivalently, if every element $x \in G$ normalizes some open, compact subgroup U of G (depending on x). This article is devoted to the study of uniscalar p -adic Lie groups. We are interested in the question whether the existence of the open, compact subgroups U normalized by individual group elements forces the existence of an open, compact subgroup normalized by all group elements simultaneously, *i.e.*, the existence of an open, compact, normal subgroup. A counterexample shows that this need not be so if the uniscalar p -adic Lie group is not compactly generated (Section 6). For compactly generated groups however, the above question has a positive answer. Calling a topological group *pro-discrete* if its filter of identity neighbourhoods has a basis of open, compact, normal subgroups, we can even prove the following stronger assertion (Theorem 5.2):

(*) *Every compactly generated, uniscalar p -adic Lie group is pro-discrete.*

We begin our studies with a characterization of uniscalar p -adic Lie groups: a p -adic Lie group G is uniscalar if and only if $\text{Ad}(G)$ is a periodic subgroup of $\text{Aut}(\mathbb{L}(G))$

¹ This work was supported by ARC grant no. A69700321 and DFG grant Ne 413/3-1.