

# The semigroup of Fredholm operators

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**Abstract.** A morphism  $f$  in a category  $\mathcal{C}$  is (von Neumann) regular if there exists a morphism  $f'$  in  $\mathcal{C}$  with,  $ff'f=f$ . In this paper, we study regularity of morphisms in certain types of categories; in particular, we derive a necessary and sufficient condition for the composite  $fg$  of two regular morphisms to be regular. We then apply these results to the category  $\mathcal{LCS}$  of locally convex spaces and show that the subcategory  $\mathcal{F}$  of  $\mathcal{LCS}$  with Fredholm operators as morphisms is a regular category, in the sense that every morphism of  $\mathcal{F}$  is regular.

Finally we study algebraic and topological properties of the semigroup  $\mathcal{F}(X)$  of Fredholm operators on a locally convex Hausdorff space  $X$ . We introduce a new integral invariant  $k(X)$  for a topological vector space  $X$  and show that: (i)  $\mathcal{F}(X)$  is semisimple if and only if  $k(X) = 1$ ; (ii)  $\mathcal{F}(X)$  is completely semisimple and unit regular if and only if  $k(X) = 0$  and (iii)  $\mathcal{F}(X)$  is simple but not semisimple if and only if  $k(X) > 1$ . We also study some important congruences on  $\mathcal{F}(X)$  and certain subsemigroups of  $\mathcal{F}(X)$ .

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## Contents

1. Introduction and summary
  2. Regularity in categories
  3. The Fredholm category
  4. The semigroup of Fredholm operators
  5. The structure of the Fredholm semigroup
  6. Regular subsemigroups of the semigroup of Fredholm operators
- References

## 1. Introduction and summary

The concept of a Fredholm operator originated from Fredholm's famous paper [12], in which he considered the problem of solving a certain type of integral equation.