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The homomorphic images of infinite symmetric groups

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Abstract. We give a solution of De Bruijn's problem as to which homomorphic images of an infinite symmetric group $S(\kappa)$ are embeddable into $S(\kappa)$. We prove (in $\mathrm{ZF} + \mathrm{V} = \mathrm{L}$) that $S(\kappa)/S_\lambda(\kappa)$ can be embedded into $S(\kappa)$ if and only if $\lambda < \mathrm{cf}(\kappa)$. We also discuss F. Clare's problem whether $S(\kappa)/S_\kappa(\kappa)$ is a universal group. We prove that it is consistent with ZFC that $S(\omega)/S_\omega(\omega)$ is not universal. However $S(\kappa)/S_\kappa(\kappa)$ is almost universal, since the group $S_{\kappa^+}(\kappa^+)$ of all bounded permutations of κ^+ embeds into it when κ is regular.

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1. Introduction

In its long history of more than 150 years the theory of finite permutation groups has reached a high level of perfection. During this rather long period only few authors also studied infinite permutation groups. A reason for this apparent disproportion may be that the rich machinery of finite combinatorics and number theory was at hand to all mathematicians while the powerful methods of infinite combinatorics (set theory) remained unknown to most of them. Without exaggeration one might say that the use of set theory consisted in the use of the bracked notation for sets and perhaps Zorn's lemma. This situation changed with the appearance of N. G. De Bruijn's, R. McKenzie's and S. Shelah's papers where a substantial amount of infinitary combinatorics was used. In 1977 a lively development started when E. B. Rabinovič ([13], Lemma 2) and Peter Neumann obtained their results about subgroups of small index in infinite permutation groups. Here and in subsequent papers by several persons methods of set theory and model theory play a dominant role. There are still numerous unsolved problems in this area. Here we discuss two of them, a problem of N. G. De Bruijn [3] and a problem of F. Clare [4].

For any nonempty set Ω let $\mathrm{Sym}(\Omega)$ be the symmetric group on Ω, i.e. the group of all bijections from Ω onto Ω. If κ is a cardinal number then write $S(\kappa)$ for $\mathrm{Sym}(\kappa)$ and