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Carleman formulae with holomorphic kernels and their uniqueness properties

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Abstract — Carleman formulae with a holomorphic kernel and integration over a boundary set of maximum dimension are obtained. These formulae have a uniqueness property: if a limit in the formula exists, it gives exactly the function which was an integrand. The Cauchy formula and its multidimensional analogies lack this property. The Carleman formulae are proved by approximating the kernel (M. M. Lavrent'ev's method).

1. ONE-DIMENSIONAL CASE

The Carleman formulae, which allow us to reconstruct the holomorphic functions in a domain using their values on part of the domain boundary, are systematically considered in the book [2]. A number of existence theorems for the Carleman formulae with a holomorphic kernel and integration over a boundary set of maximum dimension $2n-1$, in a domain $\mathcal{D} \subset \mathbf{C}^n$, were given in Section 17 [2], and there was only a single example for $n > 1$ concerning a very special case (see Theorem 16.7 in [2]). However, there exist very simple Carleman formulae of this kind, which generalize the following fact ($n = 1$).

Let Γ be a piecewise-smooth arc whose end points belong to a circle γ centered at zero (Γ belongs to the interior of γ). Suppose that $\mathcal{D}$ is a domain bounded by Γ and part of γ, and $0 \notin \mathcal{D}$. Then for any function $\mathcal{A}_C(\mathcal{D})$ [holomorphic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$] we have the Carleman formula (see Example 3 in Section 1 [2]):

$$f(z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_\Gamma f(\zeta)\left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z}. \tag{1.1}$$

This formula can be found not only by introducing a special vanishing function in the Cauchy formula (Section 1 in [2]), but also by approximating the kernel on $\partial\mathcal{D}\setminus\Gamma$. This allowed us to find simple Carleman formulae with a holomorphic kernel which generalize (1.1) for a multidimensional case. The above method goes back to Lavrent'ev's works (e.g. [7]; the Lavrent'ev method applied to the Carleman formulae for holomorphic functions is given in [2]).

Indeed, according to the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma f(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\partial\mathcal{D}\setminus\Gamma} f(\zeta)\frac{d\zeta}{\zeta - z}. \tag{1.2}$$

For $\zeta \in \partial\mathcal{D}\setminus\Gamma$ we have

$$\frac{1}{\zeta - z} = \lim_{m\to\infty} \frac{1-(z/\zeta)^m}{\zeta - z} = \lim_{m\to\infty} \sum_{k=0}^{m-1} \frac{z^k}{\zeta^{k+1}}. \tag{1.3}$$

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