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The Fischer–Riesz equations method in the ill-posed Cauchy problem for systems with injective symbols

O. V. KAREPOV* and N. N. TARKHANOV†

Abstract — An original way of applying the Fourier series theory to investigation of solvability conditions and regularization of solutions of the Cauchy problem with data on a piece of the domain boundary for systems with injective symbols is presented.

In his early paper [11] M. M. Lavrent'ev brilliantly grasped some aspects of approximate solution of the Cauchy problem for the Laplace equation. This article got the theory of conditionally stable problems off the ground.

In our paper a constructive approach to the study of one important class of conditionally stable problems, namely, the Cauchy problems with data on a boundary subset for solutions of elliptic and more general systems, is worked out.

Let $P \in \mathrm{do}_p(E \to F)$ be a differential operator (DO) of type $E \to F$ and order p with injective symbol on an open set $X \subset \mathbb{R}^n$.

Here $E = X \times \mathbb{C}^k$ and $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X whose sections of the class $\mathfrak{G}$ are interpreted as columns of functions from $\mathfrak{G}(X)$, that is $\mathfrak{G}(E) = [\mathfrak{G}(X)]^k$, and for F by analogy. Thereby, DO P is given by a $(l \times k)$-matrix of scalar DO's of order $\leq p$ on X.

For an arbitrary subset $\sigma \subset X$ the space of infinitely differentiable solutions of the system $Pf = 0$ in a neighbourhood of σ will be denoted by $S(\sigma)$ or by $S_P(\sigma)$ if we want to concretize the DO P.

By the Cauchy problem for solutions of the system $Pf = 0$ we mean the following class of boundary problems.

Suppose that $O \Subset X$ is a domain with boundary of the class C^p_{loc} (in the case $p = 1$, it requires $\partial O \in C^2_{loc}$) and $B_j \in \mathrm{do}_{b_j}(E\big|_U \to G_j)$, $j = 0, 1, \dots, p-1$, is the Dirichlet system of order $(p-1)$ on ∂O. Here G_j are vector bundles in the neighbourhood U of the boundary ∂O whose ranks are supposed to be identical and equal to the rank of E (that is l).

The system of the boundary operators $\{B_j\}$, $j = 0, 1, \dots, p-1$, is used in order to define the generalized Hardy spaces $H^q_{P,B}(O)$ of solutions of the system $Pf = 0$ in O (see [16]). Namely, for $1 \leq q \leq \infty$ the space $H^q_{P,B}(O)$ consists of such solutions $f \in S(O)$ that the expressions $B_j f$, $j = 0, 1, \dots, p-1$, have weak limits $f_j \in L^q(G_j\big|_{\partial O})$ on ∂O.

The spaces $H^q_{P,B}(O)$ with norms

$$\|f\|_{H^q_{P,B}(O)} = \left(\sum_{j=0}^{p-1} \int_{\partial O} |B_j f|^q \, ds \right)^{1/q}$$

are Banach spaces and even Hilbert ones if $q = 2$.

*Krasnoyarsk State University, Krasnoyarsk, Russia.
†Institute of Physics, Krasnoyarsk, Russia.