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Recovering a potential from Cauchy data in the two-dimensional case

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Abstract. In this paper we prove that the Cauchy data for the Schrödinger equation in the two-dimensional case determines a potential from L_p (for $p > 2$) uniquely. We also obtain a linear inversion formula for smooth potentials.

Key words. Riemann–Hilbert problem, potential, Schrödinger equation, inversion formula, Cauchy data.

AMS classification. 30G20, 35Q15, 35J05.

1. Introduction

In this paper we prove that the set of Cauchy data for the equation $\Delta u + au = 0$ in the two-dimensional case determines the potential $a \in L_p$ (for $p > 2$) uniquely. We also give an inversion formula for smooth potentials with the following property: one "measurement" with a special boundary condition gives a potential a at one given point x_0. As far as we know, these results are new even for $a \in C_0^\infty$. In our approach we mainly use techniques from [2, 22] combined with the stationary phase method. Different methods and results for this kind of problems are given in [3]–[9], [11]–[13], [15]–[20].

2. Main results

Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc. We will identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via $x = (x_1, x_2) = x_1 + \mathrm{i}x_2,\ \mathrm{i}^2 = -1$, and put

$$\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2), \qquad \partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \qquad \partial_j = \frac{\partial}{\partial x_j},$$

$$D = \begin{bmatrix} 2\overline{\partial} & 0 \\ 0 & 2\partial \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & b(x) \\ a(x) & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}.$$

Here a, b and u_j are complex-valued functions. We consider the equation

$$Pu = Du + Au = 0$$

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