Random Oper. & Stoch. Eqs., Vol. 1, No. 3, pp. 279–286 (1993)

Generalized Wigner law for band random matrices

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Received for *ROSE* 20 December 1992

Abstract—We rigorously derive the limit distribution of the eigenvalues density for band random matrices whose elements have arbitrary distribution, non-zero mean and non-constant variance.

The study of qualitative behaviour of quantum systems whose classical limit is chaotic has led to the consideration of random matrices. Indeed, analytical and numerical investigations indicate that the eigenfunctions of such quantum systems mimic a Gaussian random function. Then the matrix elements of an operator on the basis of such eigenfunctions can also be considered as random variables. It follows that random matrix ensembles are widely used to describe the statistical properties of spectra of complex systems in solid state, nuclear and atomic physics, liquid theory etc [1, 2].

The simplest case is the so-called Gaussian orthogonal ensemble (GOE) which was introduced by Wigner in the fifties to describe the properties of spectra of systems which are invariant under time-reversal. The GOE consists of real symmetric matrices with independent Gaussian random variables.

The distribution of the eigenvalues of random matrices of high order is quite complicated and therefore it is very important to derive rigorous statistical results. One of the most famous results in this field is the so-called Wigner semicircle law. Namely, it has been proven by Wigner [3] that under appropriate conditions the eigenvalues density for GOE has a semicircle form where the radius of the semicircle increases with the variance of the fluctuations of the random matrix elements. After Wigner's original paper several attempts have been made to generalize Wigner's result and to prove the semicircle law under the most general conditions. The most general formulation was given in [5, 6], where the following result was proved.

Consider a real symmetric matrix Ξ of dimension n with random independent elements ξ_{ij} with zero expectation values $\langle \xi_{ij} \rangle = 0$, and variances $\langle [\xi_{pl}]^2 \rangle = \sigma^2$, $0 < \sigma^2 < \infty$, where $\langle \cdot \rangle$ denotes the average over the random matrix ensemble. Then, for $n \to \infty$, the normalized spectral function tends, with probability 1, to the semicircle law, if and only if the Lindeberg condition holds:

$$\lim_{n\to\infty} \left\langle n^{-2} \sum_{p,k=1}^{n} |\xi_{pk}|^2 \chi\left(\left|n^{-1/2}\xi_{pk}\right| > \tau\right) \right\rangle = 0$$

for every $\tau > 0$, where χ is the characteristic function.

Translated by the authors