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The asymptotics of the number of k-dimensional subspaces of minimal weight over a finite field

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Abstract—From the totality of k-dimensional subspaces of an n-dimensional vector space over a finite field, one subspace is chosen randomly. The asymptotical behaviour ($n \to \infty$) of the probability that this subspace will be of minimal weight is established.

It is well known [1] that the number of k-dimensional subspaces V_k in the n-dimensional vector space V_n over a finite field $GF(q)$ which consists of q elements (q is a degree of a prime number) is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=0}^{k-1} (q^{n-i} - 1)/(q^{k-i} - 1). \tag{1}$$

Let $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1,\ n \geqslant 0$. The number of non-zero components of the vector $v \in V_n$ is said to be the weight of the vector v. The minimal weight of a non-zero vector $v \in V_k$ is called the weight of the k-dimensional subspace V_k of the space V_n, $1 \leqslant k \leqslant n$. Later on $V_{(k|\omega)}$ denotes the k-dimensional subspace of weight ω, $V_{(k|\omega)} \subseteq V_n$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\middle|\omega\right]$ denotes the number of all k-dimensional spaces $V_{(k|\omega)}$ in the space V_n. The number $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ can be represented in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\omega \leqslant 1} \left[\begin{matrix} n \\ k \end{matrix}\middle| \omega\right] \tag{2}$$

for $k > 0$. Our goal is to estimate the contribution of the addend $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\middle|1\right]$ in the sum on the right-hand side of (2) as $n \to \infty$.

Later on the parameter $m = n - k$ and $m \geqslant 0$.

THEOREM 1. *For the integers $n \geqslant k \geqslant 1$ the number $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\middle|1\right]$ satisfies the equalities*

$$\left[\begin{matrix} n \\ k \end{matrix}\middle|1\right] = \sum_{\mu=1}^{k} (q^m - 1)^{\mu-1} \left(\left[\begin{matrix} n-\mu \\ k-\mu+1 \end{matrix}\middle|1\right] + \begin{bmatrix} n-\mu \\ k-\mu \end{bmatrix} \right), \tag{3}$$

$$\left[\begin{matrix} n \\ k \end{matrix}\middle|1\right] = \sum_{\nu=1}^{m+1} \left(\begin{bmatrix} n-\nu \\ k-1 \end{bmatrix} + \left[\begin{matrix} n-\nu \\ k-1 \end{matrix}\middle|1\right] (q^{m+1-\nu} - 1) \right), \tag{4}$$

where $\left[\begin{smallmatrix} d \\ p \end{smallmatrix}\middle|1\right] = 0$ if either $p = 0$ or $p > d$, and $0^0 = 1$ by definition.

Before the proof of Theorem 1 we give the algorithm of the construction of the $(k \times n)$ matrix B whose rows are the basis vectors of some subspace $V_k \subseteq V_n$.

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