

# **Brownian Motion**

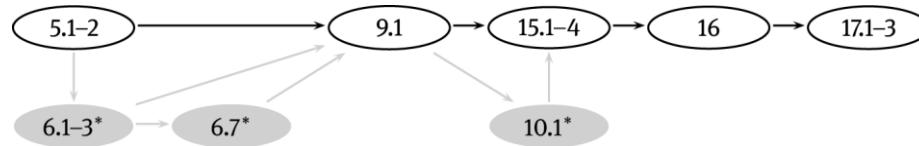
René L. Schilling / Lothar Partzsch

ISBN: 978-3-11-030729-0

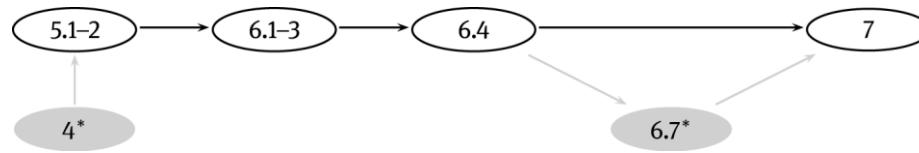
© 2014 Walter de Gruyter GmbH, Berlin/Boston

**Abbildungsübersicht / List of Figures**

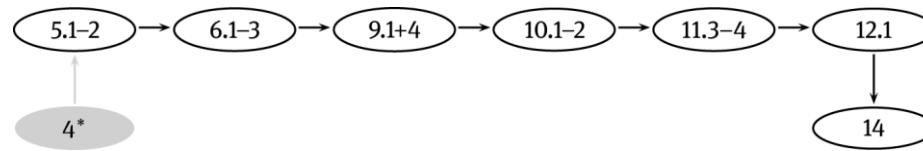
**Tabellenübersicht / List of Tables**



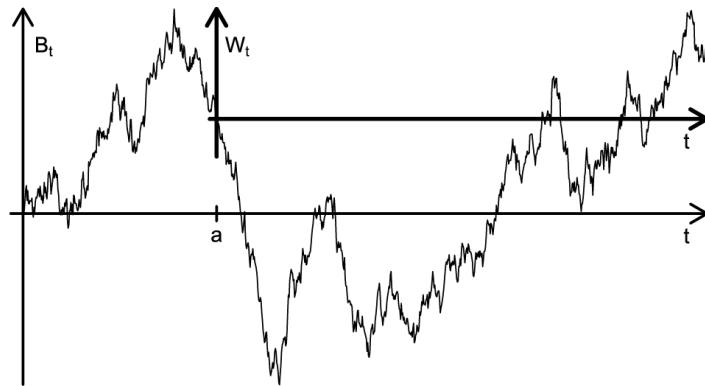
Basic stochastic calculus (C)



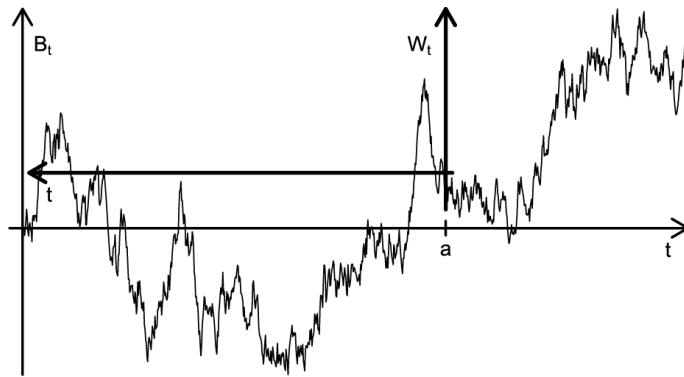
Basic Markov processes (M)



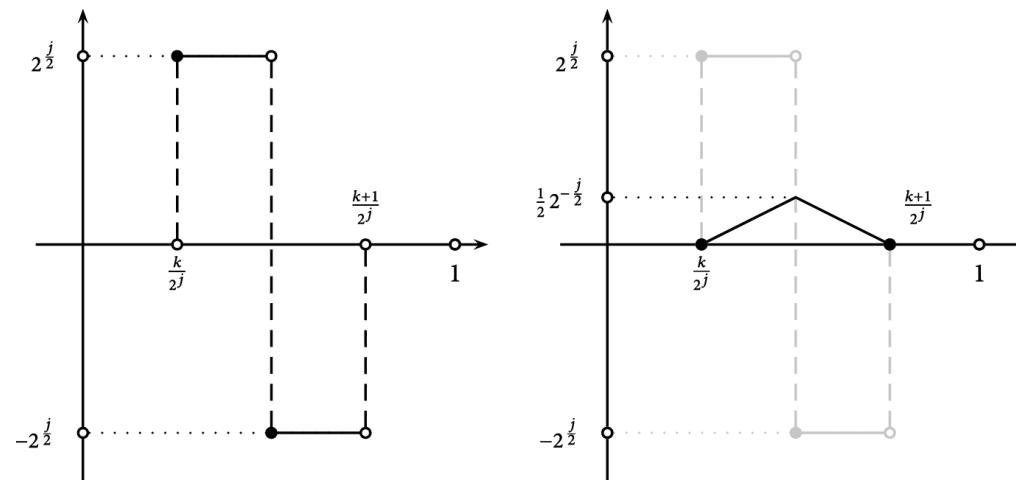
Basic sample path properties (S)



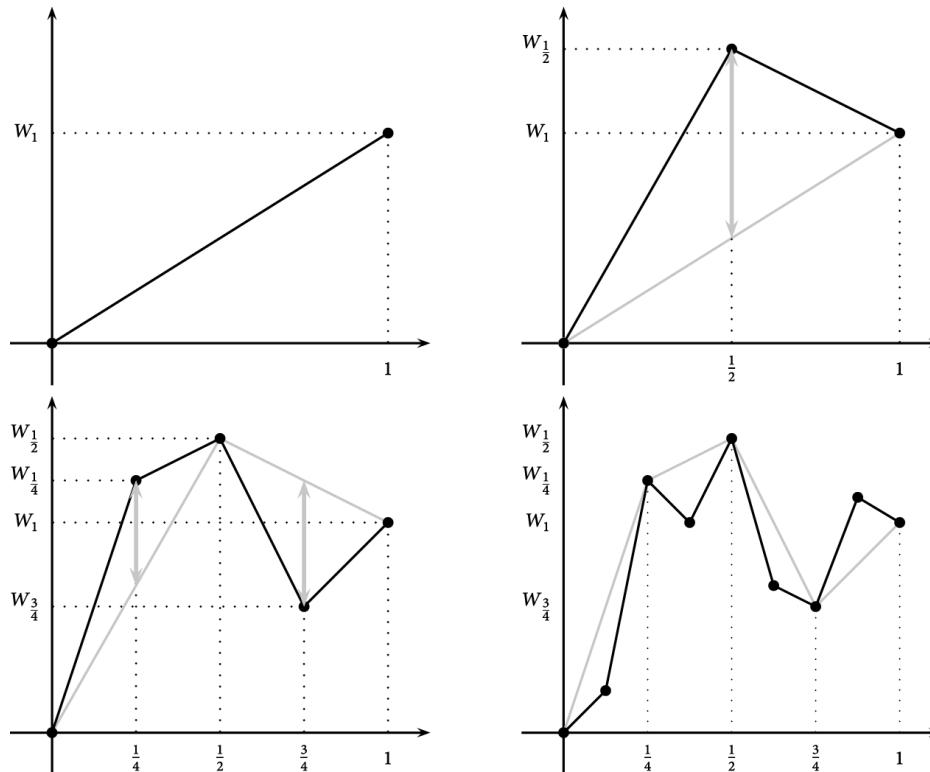
**Fig. 2.1.** Renewal at time  $a$ . The process  $W_t := B_{t+a} - B_a$ ,  $t \geq 0$ , is again a  $\text{BM}^d$ .



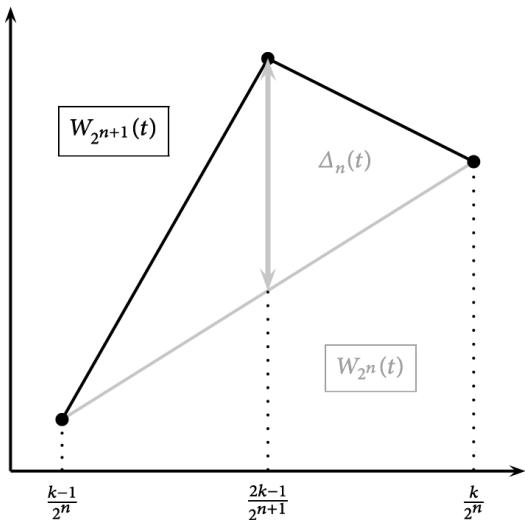
**Fig. 2.2.** Time inversion. The process  $W_t := B_{a-t} - B_t$ ,  $t \in [0, a]$ , is again a BM<sup>d</sup>.



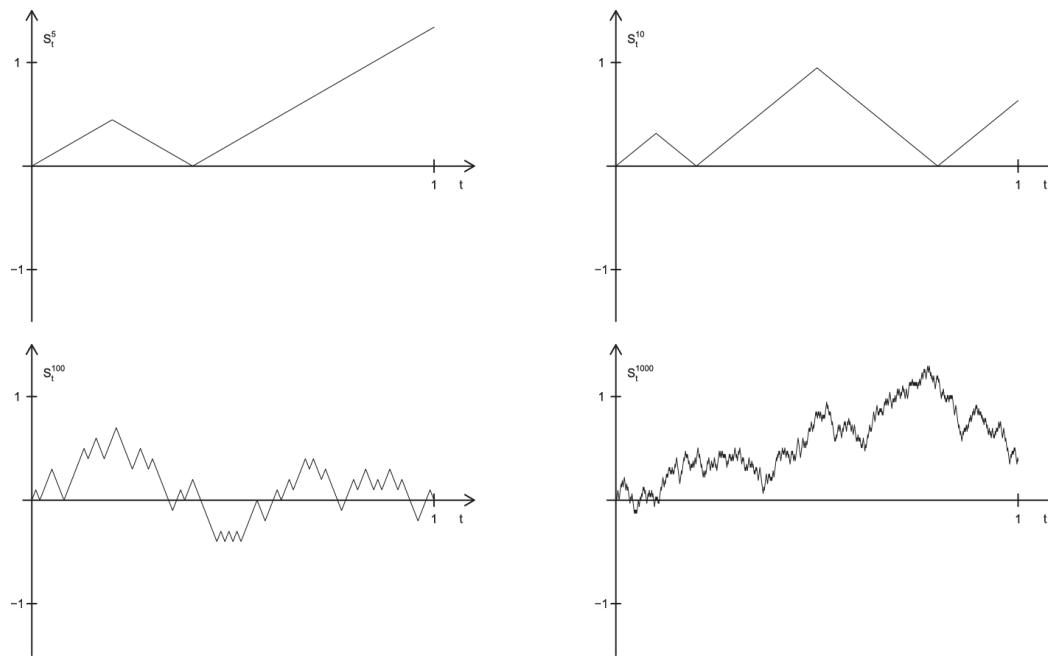
**Fig. 3.1.** The Haar functions  $H_{2^j+k}$  and the Schauder functions  $S_{2^j+k}$ .



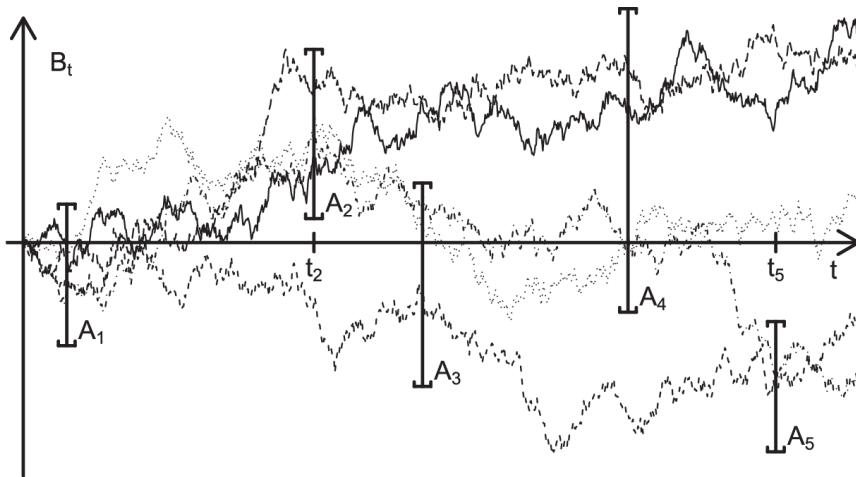
**Fig. 3.2.** The first four interpolation steps in Lévy's construction of Brownian motion.



**Fig. 3.3.** The  $n$ th interpolation step in Lévy's construction.  $\Delta_n(t) = W_{2^{n+1}}(t) - W_{2^n}(t)$



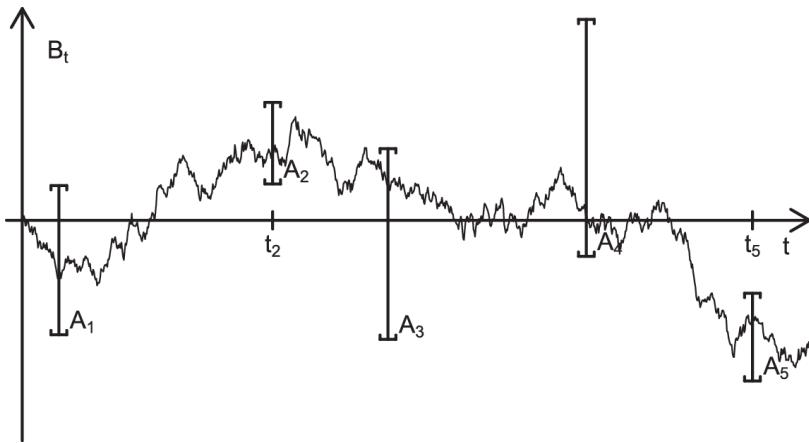
**Fig. 3.4.** The first few approximation steps in Donsker's construction:  $n = 5, 10, 100$  and  $1000$ .



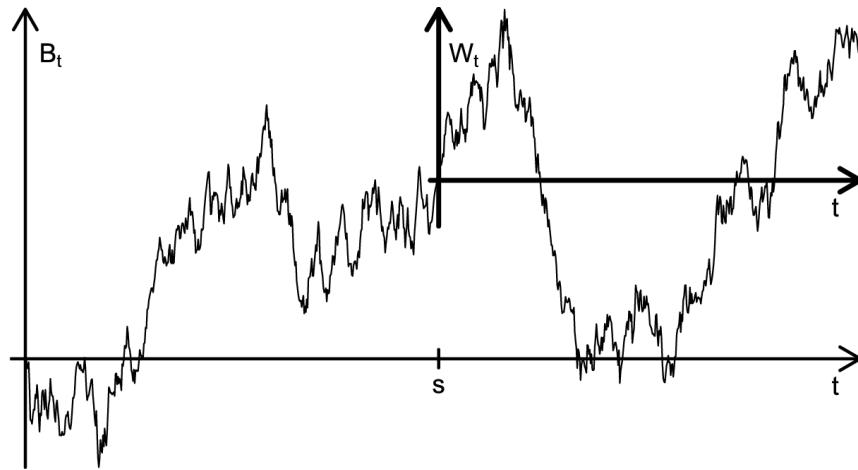
**Fig. 3.5.** Brownian paths meeting (and missing) the sets  $A_1, \dots, A_5$  at times  $t_1, \dots, t_5$

	time set $I$	paths $W(t, \omega)$	$w \in \mathcal{C}_{(0)}(I), w \mapsto S w(\cdot)$
Reflection	$[0, \infty)$	$-B(t, \omega)$	$-w(t)$
Renewal	$[0, \infty)$	$B(t + a, \omega) - B(a, \omega)$	$w(t + a) - w(a)$
Time inversion	$[0, a]$	$B(a - t, \omega) - B(a, \omega)$	$w(a - t) - w(a)$
Scaling	$[0, \infty)$	$\sqrt{c} B\left(\frac{t}{c}, \omega\right)$	$\sqrt{c} w\left(\frac{t}{c}\right)$
Proj. reflection	$[0, \infty)$	$t B\left(\frac{1}{t}, \omega\right)$	$t w\left(\frac{1}{t}\right)$

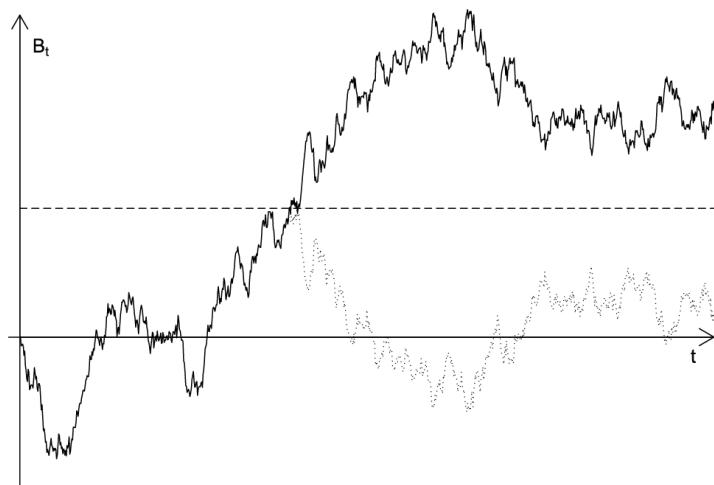
**Table 4.1.** Transformations of Brownian motion.



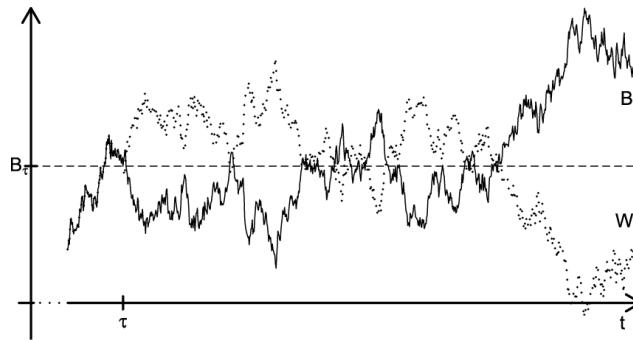
**Fig. 4.1.** A Brownian path meeting (and missing) the sets  $A_1, \dots, A_5$  at times  $t_1, \dots, t_5$



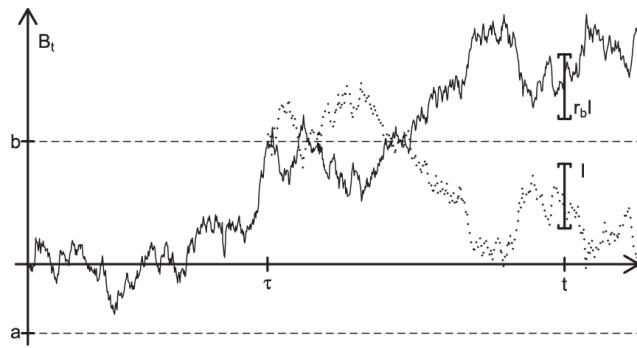
**Fig. 6.1.**  $W_t := B_{t+s} - B_s$ ,  $t \geq 0$ , is a Brownian motion in the new coordinate system with origin  $(s, B_s)$ .



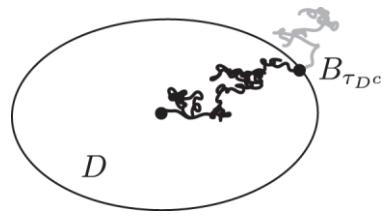
**Fig. 6.2.** Reflection upon reaching the level  $b$  for the first time.



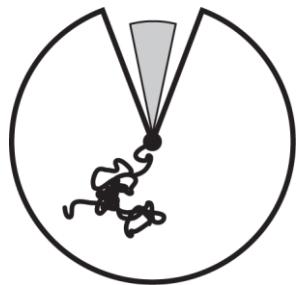
**Fig. 6.3.** Both  $B_t$  and the reflection  $W_t$  are Brownian motions.



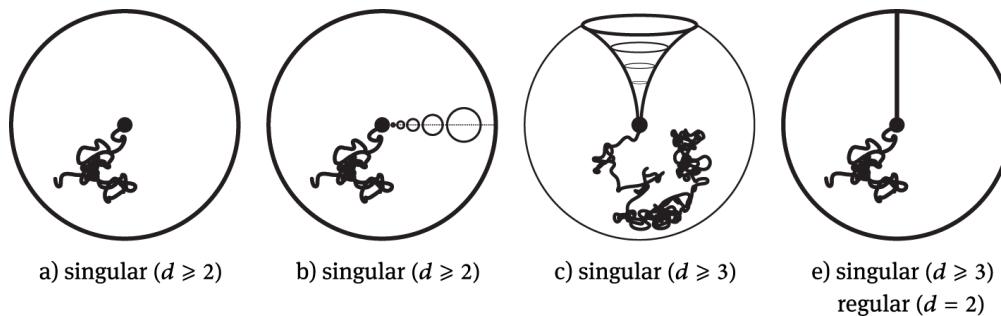
**Fig. 6.4.** A [reflected] Brownian path visiting the [reflected] interval at time  $t$ .



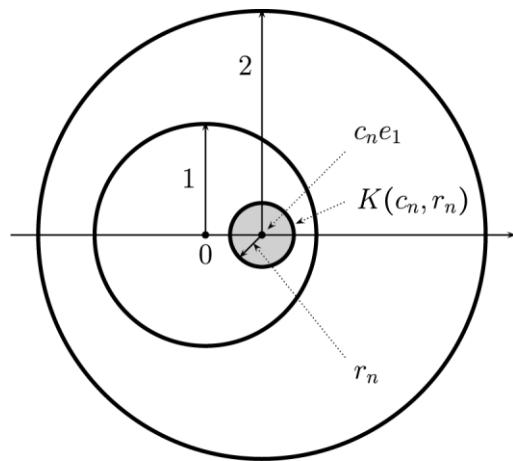
**Fig. 8.1.** Exit time and position of a Brownian motion from the set  $D$ .



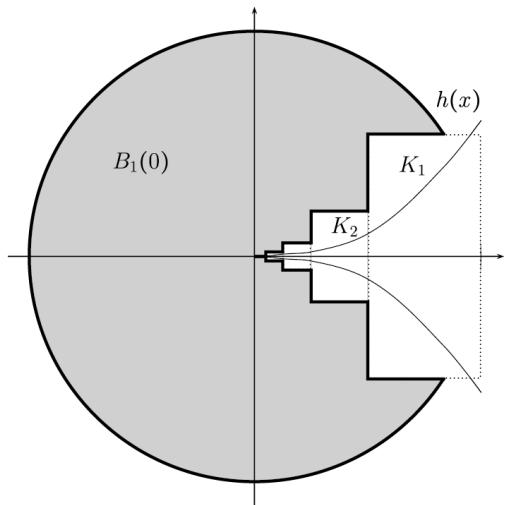
**Fig. 8.2.** The outer cone condition. A point  $x \in D$  is regular, if we can touch it with a (truncated) cone which is in  $D^c$ .



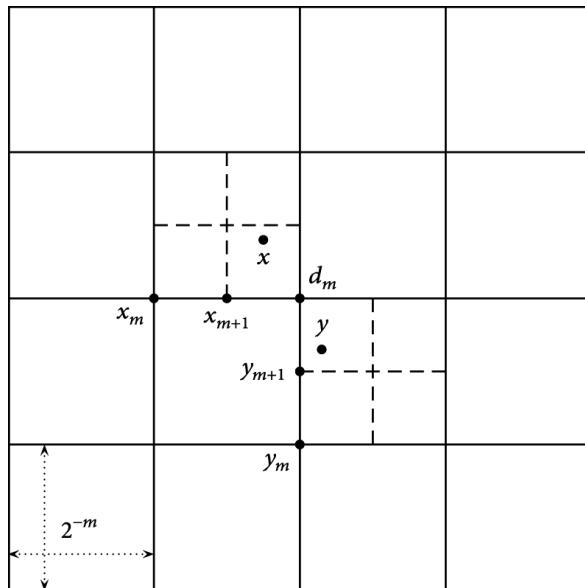
**Fig. 8.3.** Examples of singular points.



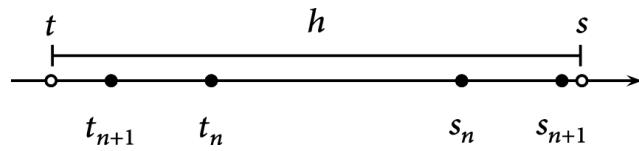
**Fig. 8.4.** Brownian motion has to exit  $\mathbb{B}(0,1)$  before it leaves  $\mathbb{B}(c_n e_1, 2)$ .



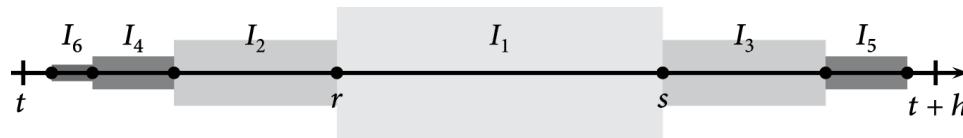
**Fig. 8.5.** A ‘cubistic’ version of Lebesgue’s spine.



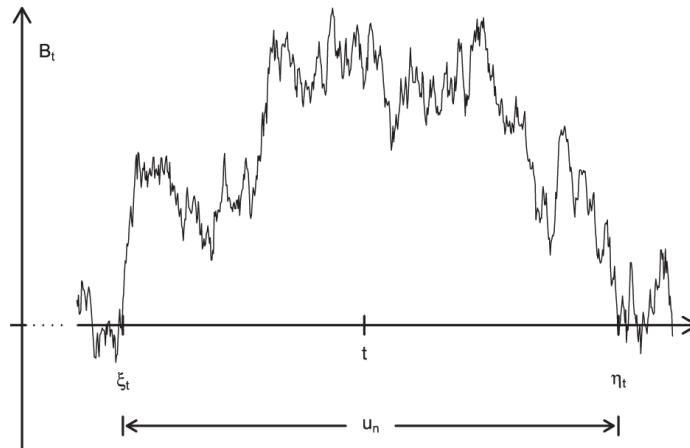
**Fig. 10.1.** Position of the points  $x$  and  $y$  and their dyadic approximations.



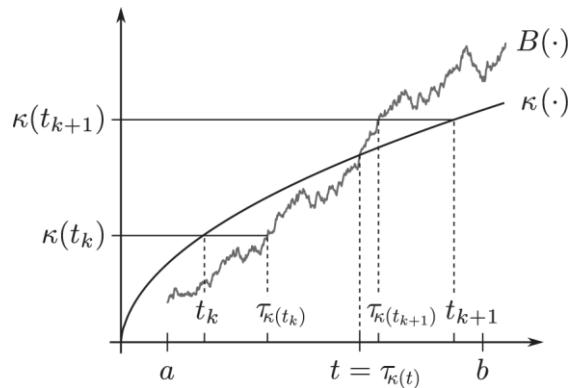
**Fig. 10.2.** Position of the points  $t_n$ ,  $t_{n+1}$  and  $s_n$ ,  $s_{n+1}$  relative to  $t$  and  $s$ .



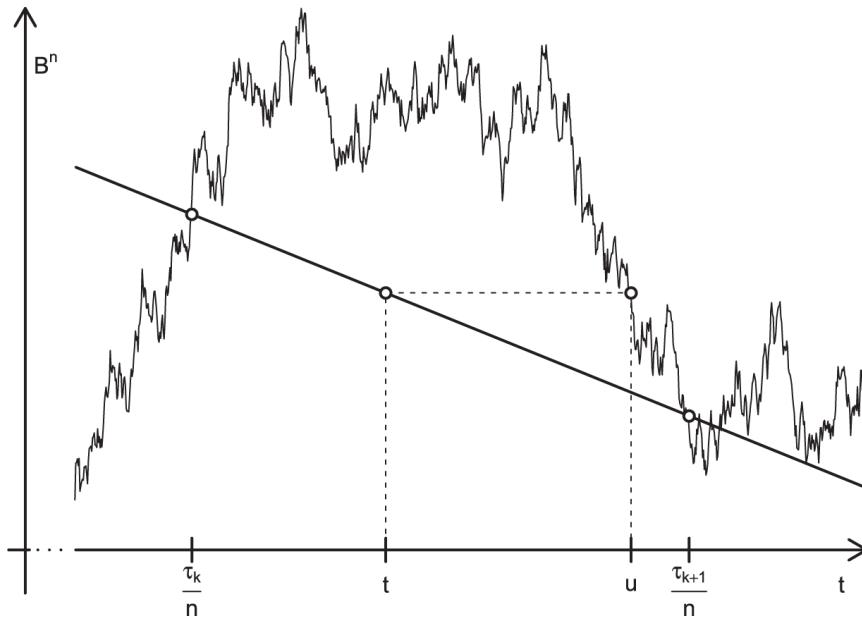
**Fig. 11.1.** Exhausting the interval  $[t, t+h]$  by non-overlapping dyadic intervals. Dots “•” denote dyadic numbers. Observe that at most two dyadic intervals of any one length can occur.



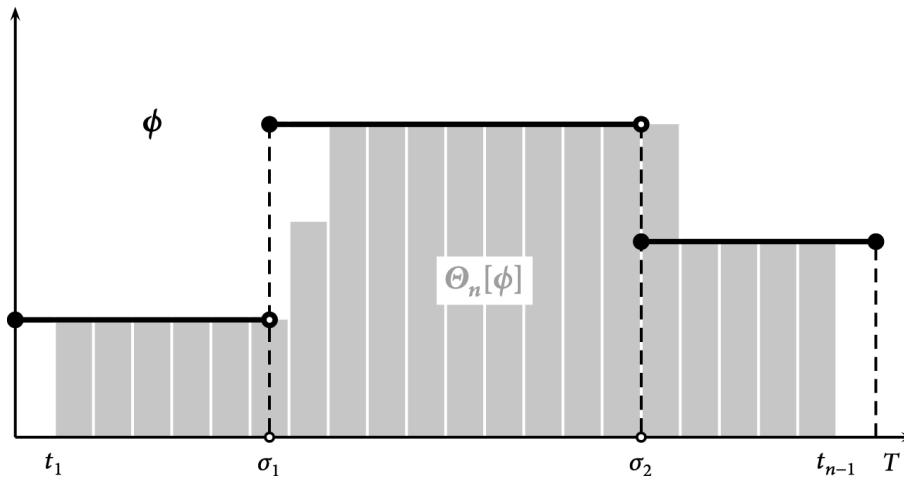
**Fig. 11.2.** A typical excursion interval:  $\xi_t$  and  $\eta_t$  are the last zero before  $t > 0$  and the first zero after  $t > 0$ .



**Fig. 12.1.** The time  $t = \tau_{\kappa(t)}$  is the first instance when the Brownian path  $[a, b] \ni s \mapsto B(s)$  reaches the graph of  $[a, b] \ni s \mapsto \kappa(s)$ .



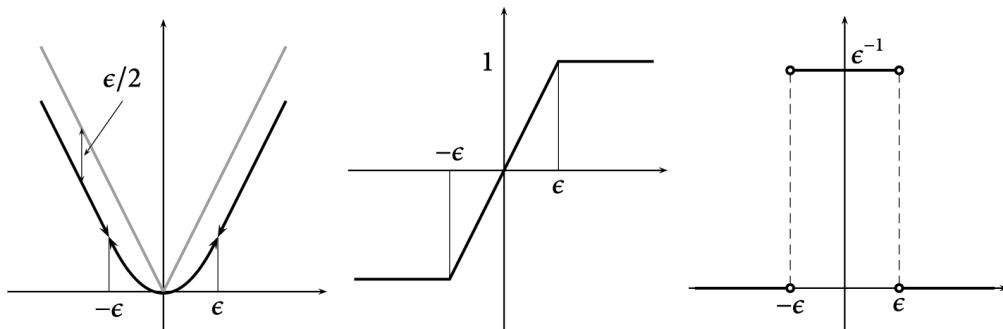
**Fig. 14.1.** For each value  $S^n(t)$  of the line segment connecting  $B^n(\tau_k/n)$  and  $B^n(\tau_{k+1}/n)$  there is some  $u$  such that  $S^n(t) = B^n(u)$ .



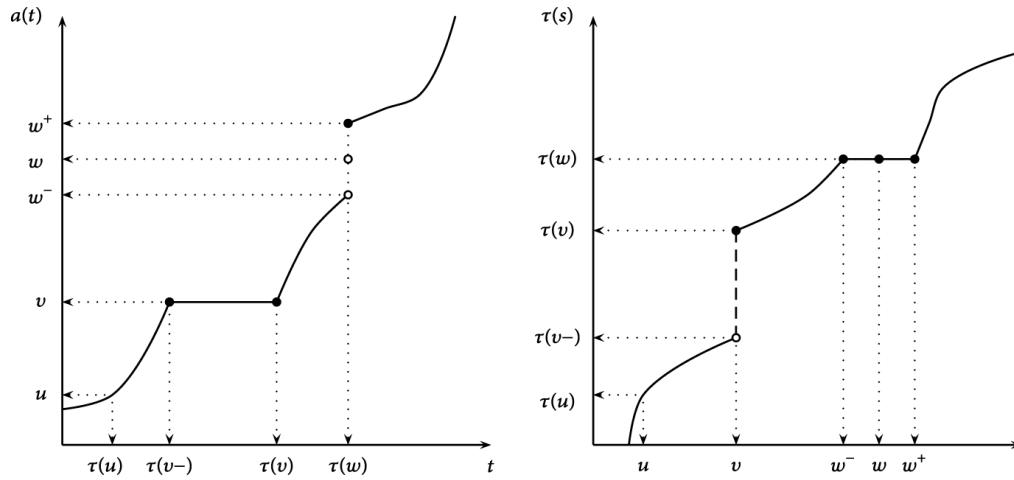
**Fig. 15.1.** Approximation of a simple function by a left-continuous simple function.

$j \neq k$	$dt$	$dB_t^j$	$dB_t^k$
$dt$	0	0	0
$dB_t^j$	0	$dt$	0
$dB_t^k$	0	0	$dt$

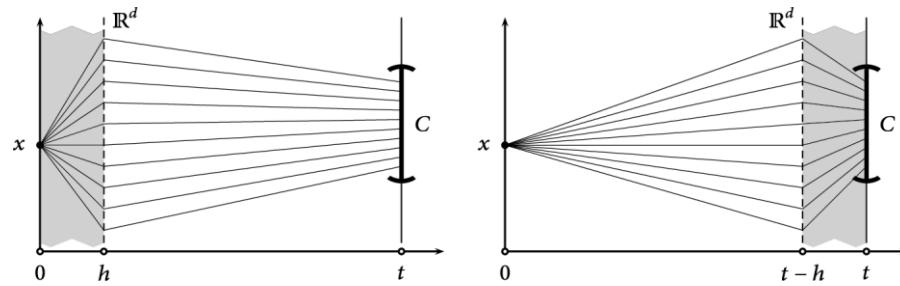
**Table 17.1.** Multiplication table for stochastic differentials.



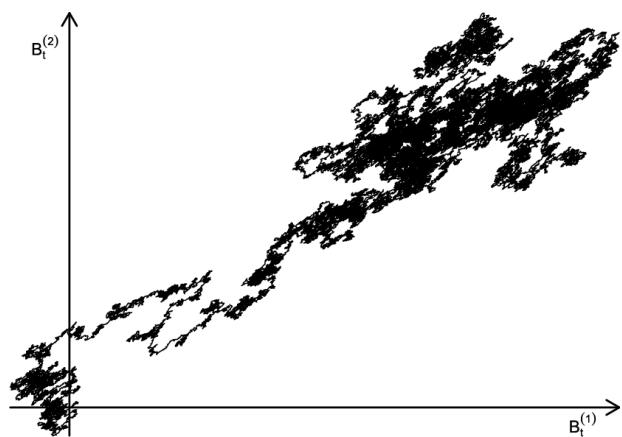
**Fig. 17.1.** Smooth approximation of the function  $x \mapsto |x|$  and its derivatives.



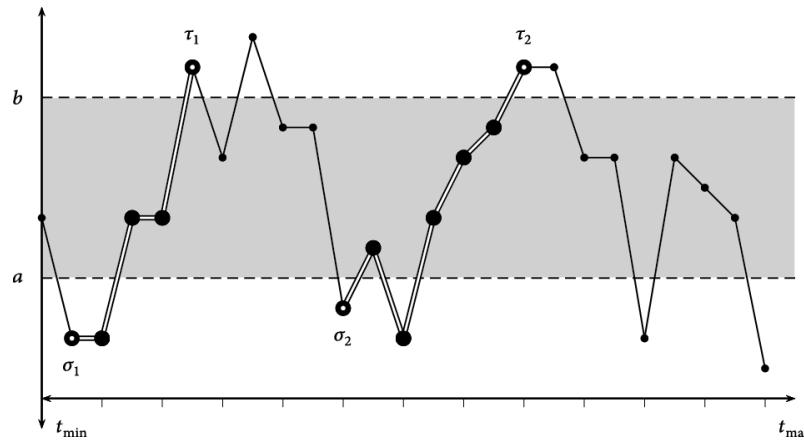
**Fig. 18.1.** An increasing right-continuous function and its generalized right-continuous inverse.



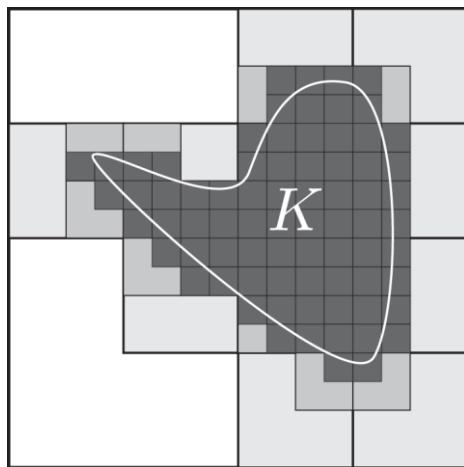
**Fig. 21.1.** Derivation of the backward and forward Kolmogorov equations.



**Fig. 22.1.** Simulation of a two-dimensional  $Q$ -Brownian motion.



**Fig. A.1.** Two upcrossings over the interval  $[a, b]$ .



**Fig. A.2.** The set  $K$  is covered by dyadic cubes of different generations.