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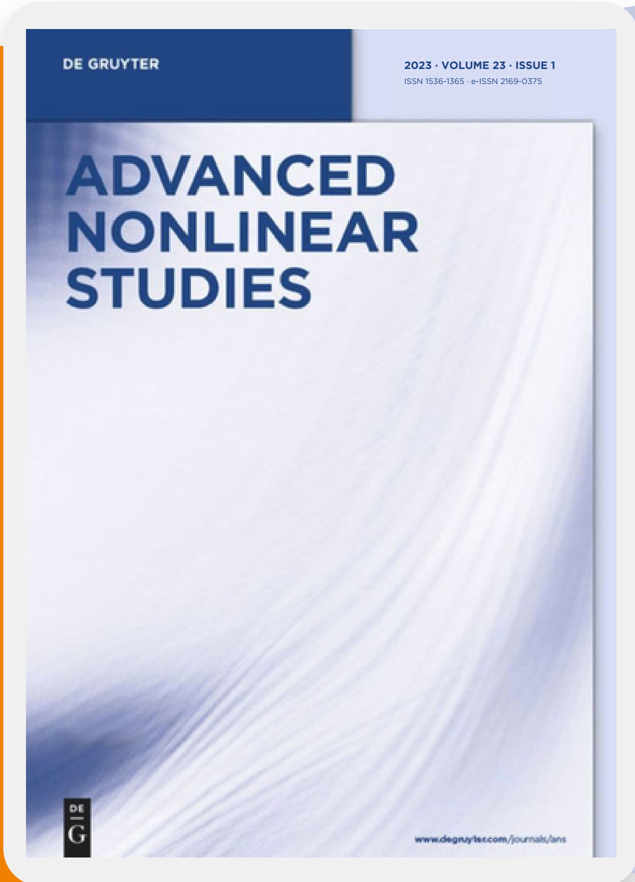

SPECIAL ISSUE:
IN HONOR OF DAVID JERISON

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Preface for the special issue in honor of David Jerison

Guozhen Lu

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To celebrate the many distinguished achievements and a remarkable career of Professor David Jerison from the Massachusetts Institute of Technology, this special issue of Advanced Nonlinear Studies is dedicated to him on the occasion of his 70th birthday.

David Jerison is an outstanding mathematician widely known for his work on partial differential equations. He is a leading figure in the areas of harmonic analysis methods to partial differential equations. He started out studying linear elliptic boundary value problems in non-smooth domains and sub-elliptic problems, especially those related to the Heisenberg group. He proved some sharp Carleman inequalities, quantitative forms of unique continuation. He then began to consider nonlinear problems. Jerison has made many fundamental and influential contributions in all these directions and his works have been followed widely by many mathematicians in the past few decades.

The classical Minkowski problem asks whether one can find a convex polyhedron given area and normal vector of each face. The corresponding inverse problem in the plane is quite easy: find the (convex) polygon given its sidelengths and angles. Jerison noticed that the Schwarz-Christoffel formula for conformal mapping solves an analogous inverse problem for harmonic measure: find the convex polygon given its angles and the harmonic measures of its sides. He generalized this to higher dimensions, solving the problem of finding a convex polyhedron given the harmonic measure and normal vector of each face. The smooth version of this problem involves a global, fully nonlinear partial differential equation relating Gauss curvature to harmonic measure. Jerison then introduced and solved a different problem that is much closer structurally to the Minkowski problem, namely, one which substitutes electrostatic capacity for volume in Minkowski's theory. Capacity can be defined as the total flux of the equilibrium potential through the boundary, and the flux is a suitably normalized harmonic measure viewed from infinity. Equivalently, capacity is the (exterior) Dirichlet energy of the equilibrium potential. There is a parallel, technically simpler, Minkowski-type problem with volume replaced by the ground state eigenvalue, which is a kind of interior Dirichlet energy of the convex body.

Inspired by seminal papers by Alt-Caffarelli, Alt-Caffarelli-Friedman, and Caffarelli, Jerison has worked for three decades on another family of nonlinear problems, free boundary problems. A major theme of this work is the parallel between free boundaries and minimal surfaces. The element that unites it with his other work is the relevance of harmonic measure. Jerison also studied an evolution equation, the Hele-Shaw equation, in which the growth of a fluid region is governed by the boundary pressure, which is none other than harmonic measure. This led him to a discrete version of this evolution known as internal diffusion-limited aggregation, a model introduced by chemists to describe corrosion and chemical polishing. Jerison's current main focus is on shapes of eigenfunctions and the curvature of their level sets. In this context, flux across level sets replaces harmonic measure.

Jerison's work on the Poincaré inequalities for Hörmander's vector fields, the CR Yamabe problem and sub-elliptic equations have influenced greatly my own research interests. My work on sharp constants and extremal functions for geometric and functional inequalities has been inspired by his pioneering work.

Jerison is a principal investigator for the Simons Collaboration on Waves in Disorder, a cross-disciplinary project based on experiments with inorganic and organic semiconductors and Bose-Einstein condensates. He serves regularly as faculty advisor to SPUR, the MIT math department's summer research program for undergraduates, as well as RSI, a high school science summer research program for students from across the world, which takes place at MIT. He is also known for his widely viewed, on-line lectures in calculus on MIT's Open Courseware and other platforms.

David Jerison was born in 1953 in Lafayette, Indiana. After college and before graduate school, he studied for a year at the University of Paris 11, at Orsay. After completing his PhD at Princeton University in 1980 under the direction of Elias Stein, he spent a year as an NSF Postdoctoral Fellow at the University of Chicago. He joined the MIT mathematics faculty in 1981, where he has remained for the rest of his career. He received a Sloan Research Fellowship and a Presidential Young Investigator Award in 1985 and delivered an invited address at the ICM in Zurich in 1994. He was selected as a Margaret MacVicar Faculty Fellow in 2004 and awarded the Bergman Prize in 2012 for his joint work with Jack Lee on the CR Yamabe problem. He received a Simons Fellowship in 2018 and a Guggenheim Fellowship in 2019. Jerison is a Fellow of the AMS and the American Academy of Arts & Sciences and served as Vice President of the AMS from 2017 to 2020.

In this special issue, we invited articles from both well-established mathematicians and young scholars. Many of them are leading researchers in the areas of harmonic analysis and partial differential equations, including some of the world's most prominent mathematicians. The wide range of topics covered in this special issue also reflect the broad scope of Jerison's research interests. I am grateful to all the contributors and reviewers for making this special issue possible.

Guozhen Lu

Editor-in-chief

Homogenization of oblique boundary value problems

Sunhi Choi, Inwon C. Kim

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We consider a nonlinear Neumann problem, with periodic oscillation in the elliptic operator and on the boundary condition. Our focus is on problems posed in half-spaces, but with general normal directions that may not be parallel to the directions of periodicity. As the frequency of the oscillation grows, quantitative homogenization results are derived. When the homogenized operator is rotation-invariant, we prove the Hölder continuity of the homogenized boundary data. While we follow the outline of Choi and Kim *(Homogenization for nonlinear PDEs in general domains with oscillatory Neumann boundary data,* Journal de Mathématiques Pures et Appliquées 102 (2014), no. 2, 419–448), new challenges arise due to the presence of tangential derivatives on the boundary condition in our problem. In addition, we improve and optimize the rate of convergence within our approach. Our results appear to be new even for the linear oblique problem.

A proof of a trace formula by Richard Melrose

Yves Colin de Verdière

doi.org/10.1515/ans-2022-0054

The goal of this article is to give a new proof of the wave trace formula proved by Richard Melrose in an impressive article. This trace formula is an extension of the Chazarain-Duistermaat-Guillemin trace formula (denoted as “CDG trace formula” in this article) to the case of a sub-Riemannian Laplacian on a 3D contact closed manifold. The proof uses a normal form constructed in previous papers, following the pioneering work of Melrose to reduce the case of the invariant Laplacian on the 3D-Heisenberg group. We need also the propagation of singularities results of the works of Ivrii, Lascar, and Melrose.

Compactness estimates for minimizers of the Alt-Phillips functional of negative exponents

Daniela De Silva, Ovidiu Savin

doi.org/10.1515/ans-2022-0055

We investigate the rigidity of global minimizers $u \geq 0$ of the Alt-Phillips functional involving negative power potentials

$$\int_{\Omega} (|\nabla u|^2 + u^{-\gamma}\chi_{\{u>0\}})\mathrm{d}x, \quad \gamma \in (0, 2),$$

when the exponent γ is close to the extremes of the admissible values. In particular, we show that global minimizers in $\mathbb{R}^n$ are one-dimensional if γ is close to 2 and $n \leq 7$, or if γ is close to 0 and $n \leq 4$.

Regularity properties of monotone measure-preserving maps

Alessio Figalli, Yash Jhaveri

doi.org/10.1515/ans-2022-0057

In this note, we extend the regularity theory for monotone measure-preserving maps, also known as optimal transports for the quadratic cost optimal transport problem, to the case when the support of the target measure is an arbitrary convex domain and, on the low-regularity end, between domains carrying certain invariant measures.

Examples of non-Dini domains with large singular sets

Carlos Kenig, Zihui Zhao

doi.org/10.1515/ans-2022-0058

Let u be a nontrivial harmonic function in a domain $D \subset \mathbb{R}^d$, which vanishes on an open set of the boundary. In a recent article, we showed that if D is a C^1-Dini domain, then, within the open set, the singular set of u, defined as

$$\{X \in \overline{D} : u(X) = 0 = |\nabla u(X)|\}$$

has finite $(d-2)$- dimensional Hausdorff measure. In this article, we show that the assumption of C^1 -Dini domains is sharp, by constructing a large class of non-Dini (but almost Dini) domains whose singular sets have infinite $\mathcal{H}^{d-2}$ -measures.

Sharp inequalities for coherent states and their optimizers

Rupert L. Frank

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We are interested in sharp functional inequalities for the coherent state transform related to the Wehrl conjecture and its generalizations. This conjecture was settled by Lieb in the case of the Heisenberg group, Lieb and Solovej for SU(2), and Kulikov for SU(1, 1) and the affine group. In this article, we give alternative proofs and characterize, for the first time, the optimizers in the general case. We also extend the recent Faber-Krahn-type inequality for Heisenberg coherent states, due to Nicola and Tilli, to the SU(2) and SU(1, 1) cases. Finally, we prove a family of reverse Hölder inequalities for polynomials, conjectured by Bodmann.

Gradient estimates and the fundamental solution for higher-order elliptic systems with lower-order terms

Ariel E. Barton, Michael J. Duffy

doi.org/10.1515/ans-2022-0064

We establish the Caccioppoli inequality, a reverse Hölder inequality in the spirit of the classic estimate of Meyers, and construct the fundamental solution for linear elliptic differential equations of order *2m* with certain lower order terms.

Propagation of symmetries for Ricci shrinkers

Tobias Holck Colding, William P. Minicozzi II

doi.org/10.1515/ans-2022-0071

We will show that if a gradient shrinking Ricci soliton has an approximate symmetry on one scale, this symmetry propagates to larger scales. This is an example of the shrinker principle which roughly states that information radiates outwards for shrinking solitons.

Linear extension operators for Sobolev spaces on radially symmetric binary trees

Charles Fefferman, Bo'az Klartag

doi.org/10.1515/ans-2022-0075

Let $1 < p < \infty$ and suppose that we are given a function f defined on the leaves of a weighted tree. We would like to extend f to a function F defined on the entire tree, so as to minimize the weighted $W^{1,p}$-Sobolev norm of the extension. An easy situation is when $p = 2$, where the harmonic extension operator provides such a function F.
In this note, we record our analysis of the particular case of a radially symmetric binary tree, which is a complete, finite, binary tree with weights that depend only on the distance from the root. Neither the averaging operator nor the harmonic extension operator work here in general. Nevertheless, we prove the existence of a linear extension operator whose norm is bounded by a constant depending solely on p.
This operator is a variant of the standard harmonic extension operator, and in fact, it is harmonic extension with respect to a certain Markov kernel determined by p and by the weights.

The Neumann problem on the domain in $\mathbb{S}3$ bounded by the Clifford torus

Jeffrey S. Case, Eric Chen, Yi Wang, Paul Yang, Po-Lam Yung

doi.org/10.1515/ans-2022-0072

In this study, the solution of the Neumann problem associated with the CR Yamabe operator on a subset Ω of the CR manifold S^3 bounded by the Clifford torus Σ is discussed. The Yamabe-type problem of finding a contact form on Ω which has zero Tanaka-Webster scalar curvature and for which Σ has a constant p-mean curvature is also discussed.

On an effective equation of the reduced Hartree-Fock theory

Ilias Chenn, Svitlana Mayboroda, Wei Wang, Shiwen Zhang

doi.org/10.1515/ans-2022-0070

We show that there is a one-to-one correspondence between solutions to the Poisson-landscape equations and the reduced Hartree-Fock equations in the semi-classical limit at low temperature. Moreover, we prove that the difference between the two corresponding solutions is small by providing explicit estimates.

Polynomial sequences in discrete nilpotent groups of step 2

Alexandru D. Ionescu, Ákos Magyar, Mariusz Mirek, Tomasz Z. Szarek

doi.org/10.1515/ans-2023-0085

We discuss some of our work on averages along polynomial sequences in nilpotent groups of step 2. Our main results include boundedness of associated maximal functions and singular integrals operators, an almost everywhere pointwise convergence theorem for ergodic averages along polynomial sequences, and a nilpotent Waring theorem. Our proofs are based on analytical tools, such as a nilpotent Weyl inequality, and on complex almost-orthogonality arguments that are designed to replace Fourier transform tools, which are not available in the noncommutative nilpotent setting. In particular, we present what we call a *nilpotent circle method* that allows us to adapt some of the ideas of the ideas of the classical circle method to the setting of nilpotent groups.

Integral inequalities with an extended poisson kernel and the existence of the extremals

Chunxia Tao, Yike Wang

doi.org/10.1515/ans-2023-0104

In this paper, we first apply the method of combining the interpolation theorem and weak-type estimate developed in [16] to derive the Hardy-Littlewood- Sobolev(HLS) inequality with an extended Poisson kernel. Using this inequality and weighted Hardy inequality, we further obtain the Stein-Weiss inequality with an ex- tended Poisson kernel. For the extremal problem of the corresponding Stein-Weiss in- equality, the presence of double-weighted exponents not being necessarily nonnegative makes it impossible to obtain the desired existence result through the usual technique of symmetrization and rearrangement. We then adopt the concentration compactness principle of double-weighted integral operator which was first used by the authors in [14] to overcome this difficulty and obtain the existence of the extremals. Finally, the regularity of positive solution of integral systems related with the extended kernel is also considered in this paper. Our regularity result also avoids the nonnegativity condition of double-weighted exponents which is a common assumption in dealing with the regularity of positive solutions of the double-weighted integral system in the literatures.

On singular solutions of Lane-Emden equation on the Heisenberg group

Juncheng Wei, Ke Wu

doi.org/10.1515/ans-2023-0084

By applying gluing method, we construct infinitely many axial symmetric singular positive solutions to the Lane-Emden equation

$$\Delta_{\mathbb{H}} u + u^p = 0, \quad \text{in } \mathbb{H}^n \backslash \{0\}$$

on the Heisenberg group $\mathbb{H}^n$, where $n > 1, Q/(Q-4) < p < p_{JL}(Q-2)$ and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$

Editorial: