ON SOME APPLICATIONS OF LIU-OWA OPERATOR

BY

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Abstract. In this paper, we introduce some new subclasses of multivalent analytic functions in the unit disc and investigate several inclusion relationships, radius problems, and some other interesting properties of \( p \)-valent functions which are defined here by means of a certain integral operator \( Q^n_{p,p}f(z) \).

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1. Introduction

Let \( A(p) \) denotes the class of functions \( f(z) \) normalized by

\[
(1.1) \quad f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),
\]

which are analytic and \( p \)-valent in the open unit disc \( E = \{z : |z| < 1\} \).

Let \( P_k(\rho) \) be the class of functions \( p(z) \) analytic in \( E \) with \( p(0) = 1 \) and

\[
(1.2) \quad \int_0^{2\pi} \left| \Re \frac{p(z)-\rho}{1-\rho} \right| d\theta \leq k\pi, \quad z = re^{i\theta},
\]

where \( k \geq 2 \) and \( 0 \leq \rho < 1 \). This class was introduced by Padmanabhan et al. (see [7]). We note that \( P_k(0) = P_k \), see Pinchuk [9], \( P_2(\rho) = P(\rho) \), the class of analytic functions with positive real part greater than \( \rho \) and \( P_2(0) = P \), the class of functions with positive real part. From (1.2) we can
easily deduce that \( p(z) \in P_k(p) \) if and only if there exists \( p_1(z), p_2(z) \in P(p) \) such that for \( z \in E \),

\[
(1.3) \quad \quad p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z).
\]

For functions \( f_j(z) \in \mathcal{A}(p) \), given by

\[
(1.4) \quad \quad f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \quad (j = 1, 2),
\]

we define the Hadamard product (or convolution) of \( f_1(z) \) and \( f_2(z) \) by

\[
(1.5) \quad \quad (f_1 \ast f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 \ast f_1)(z) \quad (z \in E).
\]

Motivated by Jung et al. [2], Liu and Owa [4] considered the linear operator \( Q_{\alpha, p}^{\beta} : \mathcal{A}(p) \rightarrow \mathcal{A}(p) \) defined as follows:

\[
(1.6) \quad Q_{\alpha, p}^{\beta} f(z) = \left( \frac{p + \alpha + \beta - 1}{p + \beta - 1} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt,
\]

for \( \alpha > 0, \beta > -1 \), and

\[
(1.7) \quad Q_{0, P}^{\beta} f(z) = f(z), \quad \text{for} \quad \alpha = 0, \beta > -1.
\]

We note that if \( f \in \mathcal{A}(p) \), then from (1.6) and (1.7) it follows that

\[
Q_{\alpha, p}^{\beta} f(z) = z^p + \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k + \beta)}{\Gamma(k + \alpha + \beta)} a^{k+p},
\]

whenever \( \alpha \geq 0 \) and \( \beta > -1 \). Using the above relation, it is easy to verify that

\[
(1.8) \quad z(Q_{\alpha, p}^{\beta} f(z))' = (p + \alpha + \beta - 1)Q_{\alpha, p}^{\beta-1} f(z) - (\alpha + \beta - 1)Q_{\alpha, p}^{\beta} f(z).
\]

For the interested readers we refer to the work done by the authors [1, 3].

Using the operator \( Q_{\alpha, p}^{\beta} \), we now define a subclasses of \( \mathcal{A}(p) \) as follows:
Definition 1.1. Assume that $\alpha \geq 0$, $\beta > -1$, $\lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $p \in \mathbb{N}$, we say that a function $f(z) \in A(p)$ is in the class $T^\alpha_{\beta,p,k}(\lambda, \rho)$ if it satisfies:

$$
\left\{ \frac{\lambda}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right)' \right\} \in P_k(\rho), \quad z \in E,
$$

where $k \geq 2$, $0 \leq \rho < p$.

Definition 1.2. Let $f \in A(p)$. Then $f \in B^\alpha_{\beta,p,k}(\lambda, \rho)$, if and only if

$$
\left\{ (1 - \lambda) \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} + \frac{\lambda}{p} \frac{(Q^{\alpha}_{\beta,p} f(z))'}{z^{p-1}} \right\} \in P_k(\rho),
$$

where $\lambda$ is a complex number, $k \geq 2$, $z \in E$, $0 \leq \rho < p$.

In the present paper, we investigate several number of inclusion relationships, radius problems, and some other interesting properties of $p$-valent functions which are defined here by means of a certain integral operator $Q^{\alpha}_{\beta,p} f(z)$.

2. Preliminaries lemmas

In this section we recall some known results.

Lemma 2.1 ([10]). If $p(z)$ is analytic in $E$ with $p(0) = 1$, and if $\lambda_1$ is a complex number satisfying $\Re(\lambda_1) \geq 0$ ($\lambda_1 \neq 0$), then $\Re \{p(z) + \lambda_1 z p'(z)\} > \delta$, $(0 \leq \delta < 1)$. Implies $\Re p(z) > \delta + (1 - \delta)(2\gamma - 1)$, where $\gamma$ is given by $\gamma = \gamma(\Re \lambda_1) = \int_0^1 (1 + t^{2\Re \lambda_1})^{-1} dt$, which is an increasing function of $\Re \lambda_1$ and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 2.2 ([11]). If $p(z)$ is analytic in $E$, $p(0) = 1$ and $\Re p(z) > \frac{1}{2}$, $z \in E$, then for any function $F$ analytic in $E$, the function $p \ast F$ takes values in the convex hull of the image of $E$ under $F$.

Lemma 2.3 ([8]). Let $p(z) = 1 + b_1 z + b_2 z^2 + \ldots \in P(\rho)$. Then $\Re p(z) \geq 2\rho - 1 + \frac{2(1 - \rho)}{1 + |z|}$.
3. Main results

**Theorem 3.1.** Assume that $\alpha \geq 0$, $\beta > -1$, $\lambda \in \mathbb{C}^* = \mathbb{C}\{0\}$, $p \in \mathbb{N}$. Let $f \in T^\alpha_{\beta,p,k}(\rho_1)$, $g \in T^\alpha_{\beta,p,k}(\rho_2)$, and let $F = f * g$. Then $F \in T^\alpha_{\beta,p,k}(\rho_3)$, where

$$\rho_3 = 1 - 4(1 - \rho_1)(1 - \rho_2) \left[ 1 - \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{rac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \frac{1}{1 + u} \, du \right].$$

**Proof.** Since $f \in T^\alpha_{\beta,p,k}(\rho_1)$, it follows that

$$H(z) = \left\{ \frac{\lambda}{p} \left( \frac{Q_{\beta,p}^{\alpha-1} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{Q_{\beta,p}^\alpha f(z)}{z^p} \right) \right\} \in P_k(\rho_1), \quad z \in E,$$

and so using identity (1.8) in the above equation, we have

$$Q_{\beta,p}^\alpha f(z) = \frac{p(p + \alpha + \beta - 1)}{\lambda} z^{p - \frac{p(p + \alpha + \beta - 1)}{\lambda}} \int_0^z t^{rac{p(p + \alpha + \beta - 1)}{\lambda} - 1} H(t) \, dt,$$

Similarly

$$Q_{\beta,p}^\alpha g(z) = \frac{p(p + \alpha + \beta - 1)}{\lambda} z^{p - \frac{p(p + \alpha + \beta - 1)}{\lambda}} \int_0^z t^{rac{p(p + \alpha + \beta - 1)}{\lambda} - 1} H^*(t) \, dt,$$

where $H^* \in P_k(\rho_2)$. Using (3.2) and (3.3), we have

$$Q_{\beta,p}^\alpha F(z) = \frac{p(p + \alpha + \beta - 1)}{\lambda} z^{p - \frac{p(p + \alpha + \beta - 1)}{\lambda}} \int_0^z t^{rac{p(p + \alpha + \beta - 1)}{\lambda} - 1} (H \ast H^*) \, dt,$$

where

$$Q(z) = \left( \frac{k}{4} + \frac{1}{2} \right) q_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) q_2(z),$$

$$H^*(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1^*(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2^*(z),$$

Now

$$H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z),$$
where \( h_i \in P(\rho_1) \) and \( h^*_i \in P(\rho_2), i = 1, 2. \)

Since \( P_i^* = \frac{h_i^*(z) - \rho_2}{2(1 - \rho_2)} + \frac{1}{2} \in P(\rho_2), i = 1, 2, \) we obtain that \((h_i * p_i^*) \in P(\rho_2), \) by using Herglotz formula.

Thus \((h_i * h_i^*) \in P(\rho_3), \) with

\[
(3.7) \quad \rho_3 = 1 - 2(1 - \rho_1)(1 - \rho_2).
\]

Using (3.4), (3.5), (3.6), (3.7) and Lemma 2.3, we have

\[
\Re q_i(z) = \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{\frac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \Re \{(h_i * h_i^*)(uz)\} \, du
\]

\[
\geq \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{\frac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \left(2\rho_3 - 1 + \frac{2(1 - \rho_3)}{1 + u|z|}\right) \, du
\]

\[
= 1 - 4(1 - \rho_1)(1 - \rho_2) \left(1 - \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{\frac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \, du\right).
\]

From this we conclude that \( F \in T_{\beta,p,k}^\alpha(\rho_3) \) where \( \rho_3 \) is given by (3.1).

We discuss the sharpness as follows:

We take

\[
H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1 + (1 - 2\rho_1)z}{1 - z} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1 - (1 - 2\rho_1)z}{1 + z},
\]

\[
H^*(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1 + (1 - 2\rho_2)z}{1 - z} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1 - (1 - 2\rho_2)z}{1 + z}.
\]

Since

\[
\left(\frac{1 + (1 - 2\rho_1)z}{1 - z}\right) * \left(\frac{1 + (1 - 2\rho_2)z}{1 - z}\right) = 1 - 4(1 - \rho_1)(1 - \rho_2) + \frac{4(1 - \rho_1)(1 - \rho_2)}{1 - z}.
\]

It follows from (3.5) that

\[
q_i(z) = \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{\frac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \cdot \left\{1 - 4(1 - \rho_1)(1 - \rho_2) + \frac{4(1 - \rho_1)(1 - \rho_2)}{1 - z}\right\} \, du
\]

\[
\rightarrow 1 - 4(1 - \rho_1)(1 - \rho_2) \left[1 - \frac{p(p + \alpha + \beta - 1)}{\lambda} \int_0^1 u^{\frac{p(p + \alpha + \beta - 1)}{\lambda} - 1} \, du\right]
\]

as \( z \rightarrow -1. \)
This completes the proof. \( \square \)

**Theorem 3.2.** Let \( f(z) \in A(p) \), \( \lambda \in \mathbb{C} \) with \( \Re\lambda > 0 \) and define the one parameter integral operator \( J_c(c > -p) \) by

\[
J_c f(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt \quad (f \in A(p); \quad c > -p).
\]

If

\[
(1 - \lambda) \frac{Q_{\beta,p}^\alpha J_c f(z)}{z^p} + \lambda \frac{Q_{\beta,p}^\alpha f(z)}{z^p} \in P_k(\rho),
\]
then \( \frac{Q_{\beta,p}^\alpha J_c f(z)}{z^p} \in P_k(\delta), z \in E, \) where

\[
\delta = \rho + (1 - \rho)(2\gamma_1 - 1),
\]
and \( \gamma_1 = \int_0^1 (1 + t^{\Re\frac{\lambda}{c+p}})^{-1} dt. \)

**Proof.** First of all it follows from the Definition 3.8 that

\[
z(Q_{\beta,p}^\alpha J_c f(z))' = (c + p)Q_{\beta,p}^\alpha f(z) - cQ_{\beta,p}^\alpha J_c f(z).
\]

Let

\[
(1 - \lambda) \frac{Q_{\beta,p}^\alpha J_c f(z)}{z^p} + \lambda \frac{Q_{\beta,p}^\alpha f(z)}{z^p} = \begin{cases} h(z) = (k + \frac{1}{2})h_1(z) - \frac{k}{2}h_2(z). \end{cases}
\]

Then, the hypothesis (3.9) in conjunction with (3.11) would yield

\[
\left\{ (1 - \lambda) \frac{Q_{\beta,p}^\alpha J_c f(z)}{z^p} + \lambda \frac{Q_{\beta,p}^\alpha f(z)}{z^p} \right\} = \left\{ h(z) + \frac{\lambda zh'(z)}{c + p} \right\} \in P_k(\rho), \text{ for } z \in E.
\]

Consequently \( \{ h(z) + \frac{\lambda zh'(z)}{c + p} \} \in P(\rho), i = 1, 2, 0 \leq \rho \leq p \) and \( z \in E. \) Using Lemma 2.1 with \( \lambda_1 = \frac{k}{(c + p)}, \) we have \( \Re\{h_i(z)\} > \delta, \) where \( \delta \) is given by (3.10), and the proof is complete. \( \square \)

**Theorem 3.3.** Let \( f \in T_{3,\beta,p,k}^\alpha(\lambda, \rho), \) and let \( \phi \in C(p), \) where \( C(p) \) is the class of \( p \)-valent convex functions. Then \( \phi \ast f \in T_{3,\beta,p,k}^\alpha(\lambda, \rho). \)
Proof. Let $F = \phi \ast f$. Then, we have
\[
\left\{ \frac{\lambda}{p} \left( \frac{Q^{\alpha-1}_{\beta,p} F(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{Q^{\alpha}_{\beta,p} F(z)}{z^p} \right) \right\} = \frac{\phi(z)}{z^p} \ast G(z),
\]
where
\[
G(z) = \left\{ \frac{\lambda}{p} \left( \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right) \right\} \in P_k(\rho).
\]
Therefore, we have $\frac{\phi(z)}{z^p} \ast G(z) = (\frac{k}{4} + \frac{1}{2})(p - \rho)(\frac{\phi(z)}{z^p} \ast g_1(z)) + \rho - (\frac{k}{4} - \frac{1}{2})(p - \rho)(\frac{\phi(z)}{z^p} \ast g_2(z)) + \rho$, $g_1, g_2 \in P$. Since $\phi \in C(p)$, $\text{Re} \{ \frac{\phi(z)}{z^p} \} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi \ast f \in T^\alpha_{\beta,p,k}(\lambda, \rho)$.

**Theorem 3.4.** For $0 \leq \lambda_2 < \lambda_1$, $T^\alpha_{\beta,p,k}(\lambda_1, \rho) \subset T^\alpha_{\beta,p,k}(\lambda_2, \rho)$.

Proof. For $\lambda_2 = 0$, the proof is immediate. Let $\lambda_2 > 0$ and $f \in T^\alpha_{\beta,p,k}(\lambda_1, \rho)$. Then
\[
\left\{ \frac{\lambda_2}{p} \left( \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^p} \right) + \frac{p - \lambda_2}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right) \right\} = \frac{\lambda_1}{p} \left( \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^p} \right) + \frac{p - \lambda_1}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right)
\]
\[
+ \left( 1 - \frac{\lambda_2}{\lambda_1} \right) \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right) = (1 - \frac{\lambda_2}{\lambda_1})H_1(z) + \frac{\lambda_2}{\lambda_1}H_2(z), \quad H_1, H_2 \in P_k(\rho).
\]
Since $P_k(\rho)$ is a convex set, see [6], we conclude that $f \in T^\alpha_{\beta,p,k}(\lambda_2, \rho)$, for $z \in E$.

**Theorem 3.5.** Let $f \in T^\alpha_{\beta,p,k}(0, \rho)$. Then $f \in T^\alpha_{\beta,p,k}(\lambda, \rho)$, for $|z| < r_\lambda = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}, \quad \lambda \neq \frac{1}{2}, \quad 0 < \lambda < 1$.

Proof. Let $\Psi_\lambda(z) = (1 - \lambda)\frac{z^p}{(1 - z)^p} + \lambda\frac{z^p}{1 - z} = z^p + \sum_{n=2}^{\infty} (1 + (n - 1)\lambda)z^{n+p-1}$. $\Psi_\lambda \in C(p)$ for $|z| < r_\lambda = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}, \quad \lambda \neq \frac{1}{2}, \quad 0 < \lambda < 1$.

We can write
\[
\left\{ \frac{\lambda}{p} \left( \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^p} \right) + \frac{p - \lambda}{p} \left( \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} \right) \right\} = \Psi_\lambda(z) \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p}.
\]
Applying Theorem 3.3, we see that $f \in T^\alpha_{\beta,p,k}(\lambda, \rho)$ for $|z| < r_\lambda$.

Now we study the interesting properties of the class $B^\alpha_{\beta,p,k}(\lambda, \rho)$.
Theorem 3.6. Let \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \). Then \( \mathcal{B}^{\alpha}_{\beta,p,k}(\lambda, \rho) \subset \mathcal{B}^{\alpha}_{\beta,p,k}(0, \rho_4) \), where \( \rho_4 \) is given by

\[
\rho_4 = \rho + (1 - \rho)(2\gamma_2 - 1),
\]

and \( \gamma_2 = \int_0^1 (1 + t^\Re(\lambda/2))^{-1} dt \).

Proof. Let \( f \in \mathcal{B}^{\alpha}_{\beta,p,k}(\lambda, \rho) \) and set

\[
(3.14) \quad \lambda, p \left( \begin{array}{c}
Q_{z_p}^{p} f(z) \\
-p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z).
\end{array} \right.
\]

Then \( p(z) \) is analytic in \( E \) with \( p(0) = 1 \). By a simple computation, we have

\[
\left\{ (1 - \lambda) \frac{Q_{z_p}^{p} f(z)}{z_p} + \frac{\lambda}{p} \frac{(Q_{z_p}^{p} f(z))'}{z_p} \right\} = \{ p(z) + \frac{1}{p} \lambda |z| \}
\]

So \( \{ p(z) + \frac{1}{p} \lambda |z| \} \in \mathcal{P}_k(\rho) \) for \( z \in E \). This implies that \( \{ p_i(z) + \frac{1}{p} \lambda |z| \} > \rho \), \( i = 1, 2 \). Using Lemma 2.1, we see that \( \Re \{ p_i(z) \} > \rho_4 \), where \( \rho_4 \) is given by (3.13). Consequently \( p \in \mathcal{P}_k(\rho_4) \) for \( z \in E \), and the proof is complete.

Now we take the converse case of Theorem 3.6.

Theorem 3.7. Let \( f \in \mathcal{B}^{\alpha}_{\beta,p,k}(0, \rho) \) for \( z \in E \). Then \( f \in \mathcal{B}^{\alpha}_{\beta,p,k}(\lambda, \rho) \) for \( |z| < R(\lambda,p) \), where

\[
(3.15) \quad \Re \{ \lambda, \rho \} = \frac{p}{|\lambda| + \sqrt{|\lambda|^2 + p}}.
\]

Proof. Set \( \frac{Q_{z_p}^{p} f(z)}{z_p} = (p - \rho)h(z) + \rho, \) \( h \in \mathcal{P}_k \). Now proceeding as in Theorem 3.6., we have

\[
\left\{ (1 - \lambda) \frac{Q_{z_p}^{p} f(z)}{z_p} + \frac{\lambda}{p} \frac{(Q_{z_p}^{p} f(z))'}{z_p} \right\} = (p - \rho) \left\{ h(z) + \frac{\lambda h'_{z_p}(z)}{p} \right\}
\]

(3.16) \( = (p - \rho) \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ h_1(z) + \frac{\lambda h'_{z_p}(z)}{p} \right\} - \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ h_2(z) + \frac{\lambda h'_{z_p}(z)}{p} \right\} \right),
\]

where we have used (1.3) and \( h_1, h_2 \in P, z \in E \). Using the following well known estimates, see [5], \( |z h'(z)| \leq \frac{2r}{1-r^2} \Re \{ h_i(z) \}, \) \( (|z| = r < 1), i = 1, 2, \)

we have \( \Re \{ h_i(z) + \frac{1}{p} \lambda |z| \} \geq \Re \{ h_i(z) - \frac{1}{p} |z| \} \), \( \Re \{ h_i(z) \} \geq \Re \{ h_i(z) \} \{ 1 - \frac{2|\lambda|}{p} \} \). The right hand side of this inequality is positive if \( r < R(\lambda,p) \),
where $R(\lambda, p)$ is given by (3.15). Consequently it follows from (3.16) that $f \in B^{\alpha}_{\beta,p,k}(\lambda, \rho)$ for $|z| < R(\lambda, p)$.

Sharpness of this result follows by taking $h_i(z) = \frac{1}{z^2}$ in (3.16), $i = 1, 2$. \hfill \qedsymbol

**Theorem 3.8.** For $0 \leq \lambda_2 < \lambda_1$, $B^{\alpha}_{\beta,p,k}(\lambda_1, \rho) \subset B^{\alpha}_{\beta,p,k}(\lambda_2, \rho)$.

**Proof.** For $\lambda_2 = 0$, the proof is immediate. Let $\lambda_2 > 0$ and let $f \in B^{\alpha}_{\beta,p,k}(\lambda_1, \rho)$. Then there exist two functions $H_1, H_2 \in P_k(\rho)$ such that from Definition 1.1 and Theorem 3.6, we have

$$\left\{ (1 - \lambda_1) \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} + \frac{\lambda_1}{p} \frac{(Q^{\alpha}_{\beta,p} f(z))'}{z^{p-1}} \right\} = H_1(z),$$

and $\frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} = H_2(z)$. Hence

$$\left\{ (1 - \lambda_2) \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} + \frac{\lambda_2}{p} \frac{(Q^{\alpha}_{\beta,p} f(z))'}{z^{p-1}} \right\} = \frac{\lambda_2}{\lambda_1} H_1(z) + (1 - \frac{\lambda_2}{\lambda_1}) H_2(z).$$

Since the class $P_k(\rho)$ is a convex set, see [6], it follows that the right hand side of (3.17) belong to $P_k(\rho)$ and this proves the result. \hfill \qedsymbol

**Theorem 3.9.** Let $f \in B^{\alpha}_{\beta,p,k}(\lambda, \rho)$, and let $\phi \in C(p)$, where $C(p)$ is the class of $p$-valent convex functions. Then $\phi * f \in B^{\alpha}_{\beta,p,k}(\lambda, \rho)$.

**Proof.** Let $F = \phi * F$. Then, we have

$$\left\{ (1 - \lambda) \frac{Q^{\alpha}_{\beta,p} F(z)}{z^p} + \frac{\lambda}{p} \frac{(Q^{\alpha}_{\beta,p} F(z))'}{z^{p-1}} \right\} = \phi(z) \frac{G(z)}{z^p} * G(z),$$

where

$$G(z) = \left\{ (1 - \lambda) \frac{Q^{\alpha}_{\beta,p} f(z)}{z^p} + \frac{\lambda}{p} \frac{(Q^{\alpha}_{\beta,p} f(z))'}{z^{p-1}} \right\} \in P_k(\rho).$$

Therefore, we have

$$\frac{\phi(z)}{z^p} * G(z) = \left( \frac{k}{4} + \frac{1}{2} \right) \left\{ (p - \rho) \left( \frac{\phi(z)}{z^p} * g_1(z) \right) + \rho \right\}$$

$$- \left( \frac{k}{4} - \frac{1}{2} \right) \left\{ (p - \rho) \left( \frac{\phi(z)}{z^p} * g_2(z) \right) + \rho \right\}, \ g_1, g_2 \in P.$$
Since $\phi \in C(p)$, $\Re \left\{ \frac{d(\phi)}{d_p} \right\} > \frac{1}{2}$, $z \in E$, and so using Lemma 2.2, we conclude that $F = \phi \ast F \in B_{\beta,p,k}^\alpha(\lambda, \rho)$. \hfill \Box

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