THE EXISTENCE OF QUASI REGULAR AND BI-REGULAR SELF-COMPLEMENTARY 3-UNIFORM HYPERGRAPHS

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Abstract

A k-uniform hypergraph $H = (V; E)$ is called self-complementary if there is a permutation $\sigma : V \to V$, called a complementing permutation, such that for every $k$-subset $e$ of $V$, $e \in E$ if and only if $\sigma(e) \notin E$. In other words, $H$ is isomorphic with $H' = (V; V^{(k)} - E)$. In this paper we define a bi-regular hypergraph and prove that there exists a bi-regular self-complementary 3-uniform hypergraph on $n$ vertices if and only if $n$ is congruent to 0 or 2 modulo 4. We also prove that there exists a quasi regular self-complementary 3-uniform hypergraph on $n$ vertices if and only if $n$ is congruent to 0 modulo 4.

Keywords: self-complementary hypergraph, uniform hypergraph, regular hypergraph, quasi regular hypergraph, bi-regular hypergraph.

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1. Introduction

Sachs [8] and Ringel [7] proved that a graph of order \( n \) is self-complementary if and only if \( n \) is congruent to 0 or 1 modulo 4. They also proved that a regular graph of order \( n \) is self-complementary if and only if \( n \) is congruent to 1 modulo 4.

Szymański and Wojda [9] proved that “A self-complementary 3-uniform hypergraph of order \( n \) exists if and only if \( n \) is congruent to 0 or 1 or 2 modulo 4.”

Potočnik, and Šajana [6] raised the following question strengthening Hartman’s conjecture [2, 3] about the existence of large sets of (not necessarily isomorphic) designs.

**Question [6].** Is it true that for every triple of integers \( t < k < n \) such that \( \binom{n-i}{k-i} \) is even for all \( i = 0, \ldots, t \), there exists a self-complementary \( t \)-subset-regular \( k \)-uniform hypergraph of order \( n \)?

The answer to the above question is affirmative for \( k = 2 \) and \( t = 1 \) (see [8]). The answer was proved affirmative also for the case \( k = 3 \) and \( t = 1 \) (see [6]). And in [4] it is shown that the answer to the question above is affirmative for the remaining case of 3-uniform hypergraphs, namely for the case \( k = 3, t = 2 \).

In this paper we digress a little from the case \( k = 3 \) and \( t = 1 \) to prove that a quasi-regular self-complementary 3-uniform hypergraph of order \( n \) exists if and only if \( n \geq 4 \) and \( n \) is congruent to 0 modulo 4, and a bi-regular self-complementary 3-uniform hypergraph of order \( n \) exists if and only if \( n \) is congruent to 0 or 2 modulo 4.

2. Preliminary Definitions and Results

**Definition (k-uniform hypergraph).** Let \( V \) be a finite set with \( n \) vertices. By \( V^{(k)} \) we denote the set of all \( k \)-subsets of \( V \). A \( k \)-uniform hypergraph is a pair \( H = (V; E) \), where \( E \subset V^{(k)} \). \( V \) is called the vertex set, and \( E \) the edge set of \( H \).

**Definition (Degree of a vertex).** The degree of a vertex \( v \) in a hypergraph \( H \) is the number of edges containing the vertex \( v \) and is denoted as \( d_H(v) \).

**Definition (Regular hypergraph).** A hypergraph \( H \) is said to be regular if all vertices have the same degree.

**Definition (Bi-regular hypergraph).** A hypergraph \( H \) is said to be bi-regular if there exist two distinct positive integers \( d_1 \) and \( d_2 \) such that the degree of each vertex is either \( d_1 \) or \( d_2 \).

**Definition (Quasi regular hypergraph).** A hypergraph \( H \) is said to be quasi regular if the degree of each vertex is either \( r \) or \( r - 1 \) for some positive integer \( r \).
It is clear that every quasi regular hypergraph is bi-regular.

**Definition** (Self-complementary $k$-uniform hypergraph). A $k$-uniform hypergraph $H = (V; E)$ is called self-complementary if there exists a permutation $\sigma : V \to V$, called a complementing permutation, such that for every $k$-subset $e$ of $V$, $e \in E$ if and only if $\sigma(e) \notin E$.

In other words, $H$ is isomorphic to $H' = (V; V^{(k)} - E)$.

**Definition** (Tournament). A tournament is a directed graph $(V, A)$ with the property that for all pairs of distinct vertices $u, v \in V$, either $(u, v) \in A$ or $(v, u) \in A$.

Further, a tournament is said to be self-converse if there exists a bijection $\varphi : V \to V$ such that for all distinct $u, v \in V$, we have $(u, v) \in A$ if and only if $(\varphi(u), \varphi(v)) \notin A$.

Kocay [5] proved the following result on complementing permutations of self-complementary 3-uniform hypergraphs.

**Proposition 1** [5]. A permutation $\sigma$ is a complementing permutation of a self-complementary 3-uniform hypergraph if and only if

(i) every cycle of $\sigma$ has even length, or
(ii) $\sigma$ has 1 or 2 fixed points, and the length of all other cycles is a multiple of 4.

Szymański and Wojda [9] proved the following result on the order of a self-complementary uniform hypergraph.

**Proposition 2** [9]. Let $k$ and $n$ be positive integers, $k \leq n$. A $k$-uniform self-complementary hypergraph of order $n$ exists if and only if $\binom{n}{k}$ is even.

**Remark 3.** For 3-uniform self-complementary hypergraph the Proposition 2 can be stated as “A 3-uniform self-complementary hypergraph of order $n$ exists if and only if $n \equiv 0 \text{ or } 1 \text{ or } 2 \pmod{4}$.

The following remark is obvious and hence is stated without proof.

**Remark 4.** If $H$ is a self-complementary 3-uniform hypergraph of order $n$ with complementing permutation $\sigma$, then

(i) for any vertex $v$ in $H$, $d_H(v) + d_H(\sigma(v)) = \binom{n-1}{2}$,
(ii) for any vertex $v$ in $H$, $d_H(v) = d_H(\sigma^2(v)) = d_H(\sigma^4(v)) = \cdots$ and $d_H(\sigma(v)) = d_H(\sigma^3(v)) = d_H(\sigma^5(v)) = \cdots$

Further, if $x$ is a fixed point of $\sigma$, then $d_H(x) = \frac{1}{2} \binom{n-1}{2}$.

**Lemma 5.** If $H$ is a self-complementary 3-uniform hypergraph on $n$ vertices, where $n$ is congruent to 1 modulo 4 and $n \geq 5$, then $H$ cannot be bi-regular.
Proof. Let $H$ be a self-complementary 3-uniform hypergraph on $n$ vertices where $n$ is congruent to 1 modulo 4, i.e., $n = 4m + 1$, $m \in \mathbb{N}$. Let $\sigma : V(H) \to V(H)$ be its complementing permutation. By Proposition 1, $\sigma$ necessarily has one fixed point, say $x$.

From Remark 4(ii) $d_H(x) = m(4m - 1)$. For $H$ to be bi-regular either $d_1 = m(4m - 1)$ or $d_2 = m(4m - 1)$. Without loss of generality let $d_1 = m(4m - 1)$. Since there are only two types of degrees $d_1$ and $d_2$, for any other vertex $v$, $d_v(H)$ is $d_1$ or $d_2$. By Remark 4(i) we have, $d_1 + d_2 = \frac{4m(4m - 1)}{2}$ which gives $d_2 = 2m(4m - 1) - m(4m - 1) = m(4m - 1) = d_1$. Hence $H$ cannot be bi-regular.

3. Existence of a Quasi Regular and Bi-Regular Self-Complementary 3-Uniform Hypergraph

The following theorem gives a necessary and sufficient condition on the order $n$ of a quasi regular self-complementary 3-uniform hypergraph. This theorem actually gives a construction of a quasi regular self-complementary 3-uniform hypergraph of desirable order.

Theorem 6. There exists a quasi regular self-complementary 3-uniform hypergraph of order $n$ if and only if $n \geq 4$ and $n \equiv 0 \pmod{4}$.

Proof. Let $H$ be a quasi regular self-complementary 3-uniform hypergraph on $n$ vertices such that degree of each vertex is either $r$ or $r - 1$ for some positive integer $r$.

![Figure 1. The types of triples making up the edge set of a quasi regular self-complementary 3-uniform hypergraph on $n = 4m$ vertices.](image)

Let $\sigma : V(H) \to V(H)$ be a complementing permutation of $H$. By Proposition 1, $\sigma$ has (i) every cycle of even length, or (ii) 1 or 2 fixed points and the
length of all the other cycles is a multiple of 4. By Remark 3, we know that a self-complementary 3-uniform hypergraph exists if and only if \( n \equiv 0 \pmod{4} \) or \( n \equiv 1 \pmod{4} \), or \( n \equiv 2 \pmod{4} \). Lemma 5 shows that \( n \) is not congruent to 1 modulo 4.

If \( n \equiv 2 \pmod{4} \), i.e., \( n = 4m + 2 \), \( m \in \mathbb{N} \), then either \( \sigma \) has 2 fixed points and the length of all other cycles is a multiple of 4 or \( \sigma \) has all cycles of even length.

If \( \sigma \) has 2 fixed points, then both must have the same degree and for some other vertex \( v \), \( d_H(v) \neq d_H(\sigma(v)) \) otherwise \( H \) will be regular. Since there are only two possible degrees \( r \) and \( r - 1 \), from Remark 4 we get that \( r + r - 1 = (n-2) = 2m(4m + 1) \), i.e., \( 2r - 1 = 2m(4m + 1) \), a contradiction.

If \( \sigma \) has all cycles of even length, then again we get the same contradiction.

Hence, if there exists a quasi regular self-complementary 3-uniform hypergraph on \( n \) vertices, then \( n \equiv 0 \pmod{4} \).

For the converse, we construct a quasi regular self-complementary 3-uniform hypergraph on \( n \) vertices where \( n \equiv 0 \pmod{4} \).

Let \( m \) be a positive integer such that \( n = 4m \) and \( V = V_0 \cup V_1 \cup V_2 \cup V_3 \), where \( V_i = \{v_j^i : j \in \mathbb{Z}_m\}, i \in \mathbb{Z}_4 \).

For every pairwise distinct triple \( i, i', i'' \in \mathbb{Z}_4 \) we define the following subsets of \( V(3) \):

\[
E_i = V_i^{(3)},
E_{(i,i')} = \{\{v^i_{j_1}, v^i_{j_2}, v^{i'}_{j_2}\} : j_1, j_2 \in \mathbb{Z}_m, j_1 \neq j_2\},
E_{(i,i',i'')} = \{\{v^i_{j_1}, v^{i'}_{j_2}, v^{i''}_{j_2}\} : j_1, j_2, j_2 \in \mathbb{Z}_m\}.
\]

Let us denote

\[
E = E_0 \cup E_1 \cup E_{(2,1)} \cup E_{(2,3)} \cup E_{(3,0)} \cup E_{(3,2)} \cup E_{(1,3)} \cup E_{(0,2)} \cup E_{(0,1,3)} \cup E_{(0,1,2)}.
\]

Let \( H \) be the 3-uniform hypergraph with vertex set \( V \) and edge set \( E \). Figure 1 explains the construction of the hypergraph \( H \). We show that \( H \) is quasi regular. Take any vertex \( v^i_j \).

Case (i) If \( i \in \{0, 1\} \), then the vertex \( v^i_j \) lies in \( \binom{m-1}{2} \) triples of \( E_i \), \( (m-1)m \) triples of \( E_{(i,i')} \), \( \binom{m}{2} \) triples of \( E_{(i,i',i'')} \), and \( 2m^2 \) triples of \( E_{(i,i',i'')} \). Hence, for every vertex \( v^i_j \) in \( H \) with \( i \in \{0, 1\} \), we have

\[
d_H(v^i_j) = \binom{m-1}{2} + \binom{m}{2} + m(m-1) + 2m^2 = 4m^2 - 3m + 1.
\]

Case (ii) If \( i \in \{2, 3\} \), then the vertex \( v^i_j \) lies in \( 2(m-1)m \) triples of \( E_{(i,i')} \), \( 2\binom{m}{2} \) triples of \( E_{(i,i',i'')} \), and \( m^2 \) triples of \( E_{(i,i',i'')} \). Hence for every vertex \( v^i_j \) in \( H \) with \( i \in \{2, 3\} \), we obtain
Thus $H$ is quasi regular with degrees $r = 4m^2 - 3m + 1$ and $r - 1 = 4m^2 - 3m$.

To prove that $H$ is self-complementary, we define a permutation $\phi : V \rightarrow V$ by $\phi(v_0^j) = v_1^j$, $\phi(v_1^j) = v_2^j$, $\phi(v_2^j) = v_3^j$ and $\phi(v_3^j) = v_0^j$, for all $j \in \mathbb{Z}_m$. Then $\phi$ is a complementing permutation of $H$ and $H$ is self-complementary.

In the next theorem we give a necessary and sufficient condition on the order $n$ of a bi-regular 3-uniform hypergraph to be self-complementary. In this theorem we shall use the following result by Alspach [1] on existence of a regular self-converse tournament.

**Theorem 7** (Alspach [1]). There exists a regular self-converse tournament with $n$ vertices for every odd integer $n$.

**Theorem 8.** There exists a bi-regular self-complementary 3-uniform hypergraph of order $n$ if and only if either $n \equiv 0 \ (\mod \ 4)$ or $n \equiv 2 \ (\mod \ 4)$ and $n \geq 4$.

**Proof.** Necessity follows from Lemma 5 and Remark 3. Conversely, let $n \equiv 0 \ (\mod \ 4)$. The self-complementary 3-uniform hypergraph constructed in Theorem 6 is quasi regular and hence bi-regular.

Let $n \equiv 2 \ (\mod \ 4)$. Then $n = 4m + 2 = 2k$ where $k = 2m + 1$ is odd. Let $V = V_0 \cup V_1$, where $V_i = \{v_i^j : j \in \mathbb{Z}_k\}, i \in \mathbb{Z}_2$. By Theorem 7, there exists a regular self-converse tournament $T = (\mathbb{Z}_k, A)$.

For $i \in \mathbb{Z}_2$, we define the following subsets of $V^{(3)}$:

$E_i = V_i^{(3)}$,

$E_{(i,i+1)} = \{(v_{j_1}^i, v_{j_2}^i, v_{j_1}^{i+1}) : j_1, j_2, j \in \mathbb{Z}_k, j_1, j_2, j \text{ pairwise distinct}\}$,

$E_A = \{(v_{k_1}^i, v_{k_2}^i, v_{k_1}^{i+1}) : (k_1, k_2) \in A, i \in \mathbb{Z}_2\}$.

Let

$E = E_0 \cup E_{(0,1)} \cup E_A$.

Let $H$ be the 3-uniform hypergraph with vertex set $V$ and edge set $E$. Figure 2 explains the construction of the hypergraph $H$. We show that $H$ is bi-regular. Let $v_j^i$ be an arbitrary vertex of $H$.

Case (i) If $i = 0$, then the vertex $v_0^j$ lies in $\binom{k-1}{2}$ triples of $E_0$, $(k - 1)(k - 2)$ triples of $E_{(0,1)}$ and $\frac{3(k-1)}{2}$ triples of $E_A$. Hence

$$d_H(v_0^j) = \binom{k-1}{2} + (k - 1)(k - 2) + \frac{3(k-1)}{2} = \frac{3(k-1)^2}{2}.$$
Case (ii) If $i = 1$, then the vertex $v_j^1$ lies in $\binom{k-1}{2}$ triples of $E_{(0,1)}$, $\frac{3(k-1)}{2}$ triples of $E_A$. Therefore,

$$d_H(v_j^1) = \binom{k-1}{2} + \frac{3(k-1)}{2} = \frac{k^2 - 1}{2}.$$ 

![Figure 2. The types of triples making up the edge set of a bi-regular self-complementary 3-uniform hypergraph on $n = 4m + 2$ vertices.](image)

This proves that $H$ is bi-regular with degrees $d_1 = \frac{3(k-1)^2}{2}$ and $d_2 = \frac{k^2 - 1}{2}$.

Let $\varphi : \mathbb{Z}_k \rightarrow \mathbb{Z}_k$ be an arc-reversing mapping of the tournament $T$; that is, $\varphi$ is a bijection on $\mathbb{Z}_k$ such that $\varphi(a) \notin A$ for all $a \in A$.

To prove that $H$ is self-complementary, we define a permutation $\phi : V \rightarrow V$ by $\phi(v_j^i) = v_j^{i+1}$ for $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_k$. $\phi$ interchanges the sets $E_1$ and $E_0$, and also the sets $E_{(0,1)}$ and $E_{(1,0)}$. Furthermore, for all $(k_1, k_2) \in A$ and $i \in \mathbb{Z}_2$, since $\varphi$ is arc-reversing, $\phi$ maps the triple $\{v_{k_1}^i, v_{k_2}^i, v_{k_1}^{i+1}\} \in E_A$ to the triple $\{v_{\varphi(k_1)}^i, v_{\varphi(k_2)}^i, v_{\varphi(k_1)}^{i+1}\} \notin E_A$. It follows that $\phi$ is a complementing permutation of $H$ and therefore $H$ is self-complementary.

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References


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