

# Grzegorzczk's Logics. Part I

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**Summary.** This article is the second in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([9] and [10]) concerning a logic proposed by Prof. Andrzej Grzegorzczk ([11]).

This part presents the syntax and axioms of Grzegorzczk's *Logic of Descriptions* (LD) as originally proposed by him, as well as some theorems not depending on any semantic constructions. There are both some clear similarities and fundamental differences between LD and the non-Fregean logics introduced by Roman Suszko in [15]. In particular, we were inspired by Suszko's semantics for his non-Fregean logic SCI, presented in [16].

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The notation and terminology used in this paper have been introduced in the following articles: [3], [17], [14], [2], [8], [4], [5], [1], [6], [12], [19], [21], [20], [13], [18], and [7].

## 1. THE CONSTRUCTION OF GRZEGORCZYK'S LD LANGUAGE

From now on  $k, m, n$  denote elements of  $\mathbb{N}$ ,  $i, j$  denote natural numbers,  $a, b, c$  denote objects,  $X, Y, Z$  denote sets,  $D, D_1, D_2$  denote non empty sets, and  $p, q, r, s$  denote finite sequences.

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The functor VAR yielding a finite sequence-membered set is defined by the term

(Def. 1) the set of all  $\langle 0, k \rangle$  where  $k$  is an element of  $\mathbb{N}$ .

Note that VAR is non empty and antichain-like.

A variable is an element of VAR. The functors: 'not', &, and '=' yielding finite sequences are defined by terms

(Def. 2)  $\langle 1 \rangle$ ,

(Def. 3)  $\langle 2 \rangle$ ,

(Def. 4)  $\langle 3 \rangle$ ,

respectively. The functor GRZ-ops yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5)  $\{ \text{'not'}, \&, '=' \}$ .

Let us note that the functor GRZ-ops yields a Polish language. The functor GRZ-symbols yielding a non empty, finite sequence-membered set is defined by the term

(Def. 6)  $\text{VAR} \cup \text{GRZ-ops}$ .

The functors: 'not', &, and '=' yield elements of GRZ-symbols. Now we state the proposition:

- (1) (i) 'not'  $\neq$  &, and
- (ii) 'not'  $\neq$  '=', and
- (iii) &  $\neq$  '='.

Observe that GRZ-symbols is non trivial and antichain-like.

The functor GRZ-op-arity yielding a function from GRZ-ops into  $\mathbb{N}$  is defined by

(Def. 7)  $it(\text{'not'}) = 1$  and  $it(\&) = 2$  and  $it('=') = 2$ .

The functor GRZ-arity yielding a Polish arity-function of GRZ-symbols is defined by

(Def. 8) for every  $a$  such that  $a \in \text{GRZ-symbols}$  holds if  $a \in \text{GRZ-ops}$ , then  $it(a) = \text{GRZ-op-arity}(a)$  and if  $a \notin \text{GRZ-ops}$ , then  $it(a) = 0$ .

Now we state the propositions:

- (2) (i)  $\text{GRZ-arity}(\text{'not'}) = 1$ , and
- (ii)  $\text{GRZ-arity}(\&) = 2$ , and
- (iii)  $\text{GRZ-arity}('=') = 2$ .

(3) The Polish atoms( $\text{GRZ-symbols}$ ,  $\text{GRZ-arity}$ ) = VAR. The theorem is a consequence of (2).

The functor GRZ-formula-set yielding a Polish language of GRZ-symbols is defined by the term

(Def. 9) Polish-WFF-set(GRZ-symbols, GRZ-arity).

A GRZ-formula is a Polish WFF of GRZ-symbols and GRZ-arity. One can verify that there exists a subset of GRZ-formula-set which is non empty.

Let us consider  $n$ . The functor  $x_n$  yielding a GRZ-formula is defined by the term

(Def. 10)  $\langle 0, n \rangle$ .

From now on  $\varphi, \psi, \vartheta, \eta$  denote GRZ-formulas.

Let us consider  $\varphi$ . The functor  $\neg\varphi$  yielding a GRZ-formula is defined by the term

(Def. 11) (Polish-unOp(GRZ-symbols, GRZ-arity, 'not'))( $\varphi$ ).

Let us consider  $\psi$ . The functors:  $\varphi \wedge \psi$  and  $\varphi = \psi$  yielding GRZ-formulas are defined by terms

(Def. 12) (Polish-binOp(GRZ-symbols, GRZ-arity, &))( $\varphi, \psi$ ),

(Def. 13) (Polish-binOp(GRZ-symbols, GRZ-arity, '='))( $\varphi, \psi$ ),

respectively. The functors:  $\varphi \vee \psi$  and  $\varphi \Rightarrow \psi$  yielding GRZ-formulas are defined by terms

(Def. 14)  $\neg(\neg\varphi \wedge \neg\psi)$ ,

(Def. 15)  $\varphi = (\varphi \wedge \psi)$ ,

respectively. The functor  $\varphi \Leftrightarrow \psi$  yielding a GRZ-formula is defined by the term

(Def. 16)  $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$ .

We say that  $\varphi$  is atomic if and only if

(Def. 17)  $\varphi \in$  the Polish atoms(GRZ-symbols, GRZ-arity).

We say that  $\varphi$  is negative if and only if

(Def. 18) Polish-WFF-head  $\varphi =$  'not'.

We say that  $\varphi$  is conjunctive if and only if

(Def. 19) Polish-WFF-head  $\varphi =$  &.

We say that  $\varphi$  is an equality if and only if

(Def. 20) Polish-WFF-head  $\varphi =$  '='.

Let us consider  $\varphi$ . Now we state the propositions:

(4)  $\varphi$  is atomic if and only if  $\varphi \in \text{VAR}$ .

(5)  $\varphi$  is negative if and only if there exists  $\psi$  such that  $\varphi = \neg\psi$ .

PROOF: If  $\varphi$  is negative, then there exists  $\psi$  such that  $\varphi = \neg\psi$  by (2), [12, (80)].  $\square$

- (6)  $\varphi$  is conjunctive if and only if there exists  $\psi$  and there exists  $\vartheta$  such that  $\varphi = \psi \wedge \vartheta$ .  
 PROOF: If  $\varphi$  is conjunctive, then there exists  $\psi$  and there exists  $\vartheta$  such that  $\varphi = \psi \wedge \vartheta$  by (2), [12, (82)].  $\square$
- (7)  $\varphi$  is an equality if and only if there exists  $\psi$  and there exists  $\vartheta$  such that  $\varphi = \psi = \vartheta$ .  
 PROOF: If  $\varphi$  is an equality, then there exists  $\psi$  and there exists  $\vartheta$  such that  $\varphi = \psi = \vartheta$  by (2), [12, (82)].  $\square$
- (8)  $\varphi$  is atomic or negative or conjunctive or an equality. The theorem is a consequence of (3).

Let us observe that every GRZ-formula which is atomic is also non negative and every GRZ-formula which is atomic is also non conjunctive and every GRZ-formula which is atomic is also non equality and every GRZ-formula which is negative is also non conjunctive and every GRZ-formula which is negative is also non equality and every GRZ-formula which is conjunctive is also non equality.

## 2. AXIOMS AND RULES

The functors: GRZ-axioms and LD-specific axioms yielding non empty subsets of GRZ-formula-set are defined by conditions

(Def. 21) for every  $a$ ,  $a \in \text{GRZ-axioms}$  iff there exists  $\varphi$  and there exists  $\psi$  and there exists  $\vartheta$  such that  $a = \neg(\varphi \wedge \neg\varphi)$  or  $a = (\neg\neg\varphi) = \varphi$  or  $a = \varphi = (\varphi \wedge \varphi)$  or  $a = (\varphi \wedge \psi) = (\psi \wedge \varphi)$  or  $a = (\varphi \wedge (\psi \wedge \vartheta)) = ((\varphi \wedge \psi) \wedge \vartheta)$  or  $a = (\varphi \wedge (\psi \vee \vartheta)) = (\varphi \wedge \psi \vee \varphi \wedge \vartheta)$  or  $a = (\neg(\varphi \wedge \psi)) = (\neg\varphi \vee \neg\psi)$  or  $a = (\varphi = \psi) = (\psi = \varphi)$  or  $a = (\varphi = \psi) = ((\neg\varphi) = (\neg\psi))$ ,

(Def. 22) for every  $a$ ,  $a \in \text{LD-specific axioms}$  iff there exists  $\varphi$  and there exists  $\psi$  and there exists  $\vartheta$  such that  $a = \varphi = \psi \Rightarrow (\varphi \wedge \vartheta) = (\psi \wedge \vartheta)$  or  $a = \varphi = \psi \Rightarrow (\varphi \vee \vartheta) = (\psi \vee \vartheta)$  or  $a = \varphi = \psi \Rightarrow (\varphi = \vartheta) = (\psi = \vartheta)$ ,

respectively. The functor LD-axioms yielding a non empty subset of GRZ-formula-set is defined by the term

(Def. 23)  $\text{GRZ-axioms} \cup \text{LD-specific axioms}$ .

A GRZ-rule is a relation between  $2^{\text{GRZ-formula-set}}$  and GRZ-formula-set. In the sequel  $R$ ,  $R_1$ ,  $R_2$  denote GRZ-rules.

Let us consider  $R_1$  and  $R_2$ . Note that the functor  $R_1 \cup R_2$  yields a GRZ-rule. The functors: GRZ-MP, GRZ-ConjIntro, GRZ-ConjElimL, and GRZ-ConjElimR yielding GRZ-rules are defined by terms

(Def. 24) the set of all  $\{\{\varphi, \varphi = \psi\}, \psi\}$  where  $\varphi$  is a GRZ-formula,  $\psi$  is a GRZ-formula,

(Def. 25) the set of all  $\langle \{\varphi, \psi\}, \varphi \wedge \psi \rangle$  where  $\varphi$  is a GRZ-formula,  $\psi$  is a GRZ-formula,

(Def. 26) the set of all  $\langle \{\varphi \wedge \psi\}, \varphi \rangle$  where  $\varphi$  is a GRZ-formula,  $\psi$  is a GRZ-formula,

(Def. 27) the set of all  $\langle \{\varphi \wedge \psi\}, \psi \rangle$  where  $\varphi$  is a GRZ-formula,  $\psi$  is a GRZ-formula, respectively. The functor GRZ-rules yielding a GRZ-rule is defined by

(Def. 28) for every  $a$ ,  $a \in it$  iff  $a \in \text{GRZ-MP}$  or  $a \in \text{GRZ-ConjIntro}$  or  $a \in \text{GRZ-ConjElimL}$  or  $a \in \text{GRZ-ConjElimR}$ .

A GRZ-formula sequence is a finite sequence of elements of GRZ-formula-set.

A finite GRZ-formula set is a finite subset of GRZ-formula-set. From now on  $\Gamma$ ,  $\Gamma_1$ ,  $\Gamma_2$  denote non empty subsets of GRZ-formula-set,  $\Delta$ ,  $\Delta_1$ ,  $\Delta_2$  denote subsets of GRZ-formula-set,  $P$ ,  $P_1$ ,  $P_2$  denote GRZ-formula sequences, and  $\Sigma$ ,  $\Sigma_1$ ,  $\Sigma_2$  denote finite GRZ-formula sets.

Let us consider  $\Sigma_1$  and  $\Sigma_2$ . Observe that the functor  $\Sigma_1 \cup \Sigma_2$  yields a finite GRZ-formula set. Let us consider  $\Gamma$ ,  $R$ ,  $P$ , and  $n$ . We say that  $(P, n)$  is a correct step w.r.t.  $\Gamma$ ,  $R$  if and only if

(Def. 29)  $P(n) \in \Gamma$  or there exists a finite GRZ-formula set  $Q$  such that  $\langle Q, P(n) \rangle \in R$  and for every  $q$  such that  $q \in Q$  there exists  $k$  such that  $k \in \text{dom } P$  and  $k < n$  and  $P(k) = q$ .

We say that  $P$  is  $(\Gamma, R)$ -correct if and only if

(Def. 30) for every  $k$  such that  $k \in \text{dom } P$  holds  $(P, k)$  is a correct step w.r.t.  $\Gamma$ ,  $R$ .

Let  $a$  be an element of  $\Gamma$ . One can verify that the functor  $\langle a \rangle$  yields a GRZ-formula sequence. Now we state the proposition:

(9) Let us consider an element  $a$  of  $\Gamma$ . Then  $\langle a \rangle$  is  $(\Gamma, R)$ -correct.

Let us consider  $\Gamma$  and  $R$ . Note that there exists a GRZ-formula sequence which is non empty and  $(\Gamma, R)$ -correct.

Let us consider  $\Sigma$ . We say that  $\Sigma$  is  $(\Gamma, R)$ -correct if and only if

(Def. 31) there exists  $P$  such that  $\Sigma = \text{rng } P$  and  $P$  is  $(\Gamma, R)$ -correct.

Now we state the propositions:

(10) If  $P$  is  $(\Gamma, R)$ -correct and  $P = P_1 \wedge P_2$ , then  $P_1$  is  $(\Gamma, R)$ -correct.

(11) If  $P_1$  is  $(\Gamma, R)$ -correct and  $P_2$  is  $(\Gamma, R)$ -correct, then  $P_1 \wedge P_2$  is  $(\Gamma, R)$ -correct.

(12) If  $\Sigma_1$  is  $(\Gamma, R)$ -correct and  $\Sigma_2$  is  $(\Gamma, R)$ -correct, then  $\Sigma_1 \cup \Sigma_2$  is  $(\Gamma, R)$ -correct. The theorem is a consequence of (11).

(13) If  $\Gamma \subseteq \Gamma_1$  and  $R \subseteq R_1$  and  $P$  is  $(\Gamma, R)$ -correct, then  $P$  is  $(\Gamma_1, R_1)$ -correct.

Let us consider  $\Gamma$ ,  $R$ , and  $\varphi$ . We say that  $\Gamma, R \vdash \varphi$  if and only if

(Def. 32) there exists  $P$  such that  $\varphi \in \text{rng } P$  and  $P$  is  $(\Gamma, R)$ -correct.

Let us consider  $\Delta$ . We say that  $\Gamma, R \vdash \Delta$  if and only if

(Def. 33) for every  $\varphi$  such that  $\varphi \in \Delta$  holds  $\Gamma, R \vdash \varphi$ .

Let us consider  $\Gamma, R$ , and  $\varphi$ . Now we state the propositions:

(14)  $\Gamma, R \vdash \varphi$  if and only if there exists  $\Sigma$  such that  $\varphi \in \Sigma$  and  $\Sigma$  is  $(\Gamma, R)$ -correct.

(15) If  $\varphi \in \Gamma$ , then  $\Gamma, R \vdash \varphi$ . The theorem is a consequence of (9).

Now we state the propositions:

(16) If  $\Gamma, R \vdash \Sigma$ , then there exists  $\Sigma_1$  such that  $\Sigma \subseteq \Sigma_1$  and  $\Sigma_1$  is  $(\Gamma, R)$ -correct.

PROOF: Define  $\mathcal{X}[\text{set}] \equiv$  there exists  $\Sigma_1$  such that  $\text{\$}_1 \subseteq \Sigma_1$  and  $\Sigma_1$  is  $(\Gamma, R)$ -correct.  $\mathcal{X}[\emptyset]$ . For every sets  $x, \Delta$  such that  $x \in \Sigma$  and  $\Delta \subseteq \Sigma$  and  $\mathcal{X}[\Delta]$  holds  $\mathcal{X}[\Delta \cup \{x\}]$ .  $\mathcal{X}[\Sigma]$  from [8, Sch. 2].  $\square$

(17) If  $\Gamma, R \vdash \Sigma$  and  $\langle \Sigma, \varphi \rangle \in R$ , then  $\Gamma, R \vdash \varphi$ . The theorem is a consequence of (16).

(18) If  $\Gamma, R \vdash \varphi$ , then  $\varphi \in \Gamma$  or there exists  $\Sigma$  such that  $\langle \Sigma, \varphi \rangle \in R$  and  $\Gamma, R \vdash \Sigma$ .

(19) If  $\Gamma \subseteq \Gamma_1$  and  $R \subseteq R_1$  and  $\Gamma, R \vdash \varphi$ , then  $\Gamma_1, R_1 \vdash \varphi$ .

Let us consider  $\Gamma, \varphi$ , and  $\psi$ . Now we state the propositions:

(20)  $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$  if and only if  $\Gamma, \text{GRZ-rules} \vdash \varphi$  and  $\Gamma, \text{GRZ-rules} \vdash \psi$ . The theorem is a consequence of (17).

(21) Suppose  $\Gamma, \text{GRZ-rules} \vdash \varphi$  and  $\Gamma, \text{GRZ-rules} \vdash \varphi = \psi$ . Then  $\Gamma, \text{GRZ-rules} \vdash \psi$ . The theorem is a consequence of (17).

(22) Suppose  $\Gamma, \text{GRZ-rules} \vdash \varphi$  and  $\Gamma, \text{GRZ-rules} \vdash \varphi \Rightarrow \psi$ .

Then  $\Gamma, \text{GRZ-rules} \vdash \psi$ . The theorem is a consequence of (21) and (20).

(23) If  $\Gamma, \text{GRZ-rules} \vdash \varphi \wedge \psi$ , then  $\Gamma, \text{GRZ-rules} \vdash \psi \wedge \varphi$ . The theorem is a consequence of (20).

Let us consider  $\varphi$ . We say that  $\varphi$  is GRZ-axiomatic if and only if

(Def. 34)  $\varphi \in \text{GRZ-axioms}$ .

We say that  $\varphi$  is GRZ-provable if and only if

(Def. 35)  $\text{GRZ-axioms}, \text{GRZ-rules} \vdash \varphi$ .

We say that  $\varphi$  is LD-axiomatic if and only if

(Def. 36)  $\varphi \in \text{LD-axioms}$ .

We say that  $\varphi$  is LD-provable if and only if

(Def. 37)  $\text{LD-axioms}, \text{GRZ-rules} \vdash \varphi$ .

Observe that  $\neg(\varphi \wedge \neg\varphi)$  is GRZ-axiomatic and  $(\neg\neg\varphi) = \varphi$  is GRZ-axiomatic and  $\varphi = (\varphi \wedge \varphi)$  is GRZ-axiomatic.

Let us consider  $\psi$ . Observe that  $(\varphi \wedge \psi) = (\psi \wedge \varphi)$  is GRZ-axiomatic and  $(\neg(\varphi \wedge \psi)) = (\neg\varphi \vee \neg\psi)$  is GRZ-axiomatic and  $(\varphi = \psi) = (\psi = \varphi)$  is GRZ-axiomatic and  $(\varphi = \psi) = ((\neg\varphi) = (\neg\psi))$  is GRZ-axiomatic.

Let us consider  $\vartheta$ . Observe that  $(\varphi \wedge (\psi \wedge \vartheta)) = ((\varphi \wedge \psi) \wedge \vartheta)$  is GRZ-axiomatic and  $(\varphi \wedge (\psi \vee \vartheta)) = (\varphi \wedge \psi \vee \varphi \wedge \vartheta)$  is GRZ-axiomatic and  $\varphi = \psi \Rightarrow (\varphi \wedge \vartheta) = (\psi \wedge \vartheta)$  is LD-axiomatic and  $\varphi = \psi \Rightarrow (\varphi \vee \vartheta) = (\psi \vee \vartheta)$  is LD-axiomatic and  $\varphi = \psi \Rightarrow (\varphi = \vartheta) = (\psi = \vartheta)$  is LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also LD-axiomatic and every GRZ-formula which is GRZ-axiomatic is also GRZ-provable and every GRZ-formula which is LD-axiomatic is also LD-provable and every GRZ-formula which is GRZ-provable is also LD-provable and there exists a GRZ-formula which is GRZ-axiomatic, GRZ-provable, LD-axiomatic, and LD-provable.

Now we state the proposition:

(24) Suppose  $\text{GRZ-axioms} \subseteq \Gamma$  and  $\Gamma, \text{GRZ-rules} \vdash \varphi = \psi$ .

Then  $\Gamma, \text{GRZ-rules} \vdash \psi = \varphi$ . The theorem is a consequence of (15) and (21).

### 3. PROVABILITY

Let us consider  $\varphi$  and  $\psi$ . Now we state the propositions:

(25) If  $\varphi = \psi$  is GRZ-provable, then  $\psi = \varphi$  is GRZ-provable.

(26) If  $\varphi = \psi$  is LD-provable, then  $\psi = \varphi$  is LD-provable.

Now we state the propositions:

(27) If  $\varphi = \psi$  is LD-provable and  $\psi = \vartheta$  is LD-provable, then  $\varphi = \vartheta$  is LD-provable.

The theorem is a consequence of (24), (22), and (21).

(28)  $\varphi = \varphi$  is LD-provable. The theorem is a consequence of (24) and (27).

Let us consider  $\varphi$  and  $\psi$ . We say that  $\varphi =_{\text{LD}} \psi$  if and only if

(Def. 38)  $\varphi = \psi$  is LD-provable.

One can check that the predicate is reflexive and symmetric.

Now we state the proposition:

(29) If  $\varphi =_{\text{LD}} \psi$ , then  $\neg\varphi =_{\text{LD}} \neg\psi$ . The theorem is a consequence of (21).

The scheme *BinReplace* deals with a non empty set  $\mathcal{X}$  and a binary functor  $\mathcal{F}$  yielding an element of  $\mathcal{X}$  and a binary predicate  $\mathcal{R}$  and states that

(Sch. 1) For every elements  $a, b, c, d$  of  $\mathcal{X}$  such that  $\mathcal{R}[a, b]$  and  $\mathcal{R}[c, d]$  holds  $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, d)]$

provided

- for every elements  $a, b, c$  of  $\mathcal{X}$  such that  $\mathcal{R}[a, b]$  and  $\mathcal{R}[b, c]$  holds  $\mathcal{R}[a, c]$  and

- for every elements  $a, b$  of  $\mathcal{X}$ ,  $\mathcal{R}[\mathcal{F}(a, b), \mathcal{F}(b, a)]$  and
- for every elements  $a, b, c$  of  $\mathcal{X}$  such that  $\mathcal{R}[a, b]$  holds  $\mathcal{R}[\mathcal{F}(a, c), \mathcal{F}(b, c)]$ .

Let us consider  $\varphi, \psi, \vartheta$ , and  $\eta$ .

Let us assume that  $\varphi =_{\text{LD}} \psi$  and  $\vartheta =_{\text{LD}} \eta$ . Now we state the propositions:

$$(30) \quad \varphi \wedge \vartheta =_{\text{LD}} \psi \wedge \eta.$$

PROOF: Define  $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 \wedge \$_2$ . Define  $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1 = \$_2$  is LD-provable. For every  $\varphi, \psi$ , and  $\vartheta$  such that  $\mathcal{P}[\varphi, \psi]$  and  $\mathcal{P}[\psi, \vartheta]$  holds  $\mathcal{P}[\varphi, \vartheta]$ . For every  $\varphi, \psi, \vartheta$ , and  $\eta$  such that  $\mathcal{P}[\varphi, \psi]$  and  $\mathcal{P}[\vartheta, \eta]$  holds  $\mathcal{P}[\mathcal{F}(\varphi, \vartheta), \mathcal{F}(\psi, \eta)]$  from *BinReplace*.  $\square$

$$(31) \quad \varphi =_{\text{LD}} \vartheta =_{\text{LD}} \psi =_{\text{LD}} \eta.$$

PROOF: Define  $\mathcal{F}(\text{GRZ-formula}, \text{GRZ-formula}) = \$_1 = \$_2$ . Define  $\mathcal{P}[\text{GRZ-formula}, \text{GRZ-formula}] \equiv \$_1 = \$_2$  is LD-provable. For every  $\varphi, \psi$ , and  $\vartheta$  such that  $\mathcal{P}[\varphi, \psi]$  and  $\mathcal{P}[\psi, \vartheta]$  holds  $\mathcal{P}[\varphi, \vartheta]$ . For every  $\varphi, \psi, \vartheta$ , and  $\eta$  such that  $\mathcal{P}[\varphi, \psi]$  and  $\mathcal{P}[\vartheta, \eta]$  holds  $\mathcal{P}[\mathcal{F}(\varphi, \vartheta), \mathcal{F}(\psi, \eta)]$  from *BinReplace*.  $\square$

The functor LD-IdR yielding an equivalence relation of GRZ-formula-set is defined by

(Def. 39) for every  $\varphi$  and  $\psi$ ,  $\langle \varphi, \psi \rangle \in it$  iff  $\varphi =_{\text{LD}} \psi$ .

Note that there exists a family of subsets of GRZ-formula-set which is non empty.

The functor LD-IdClasses yielding a non empty family of subsets of GRZ-formula-set is defined by the term

(Def. 40) Classes LD-IdR.

An LD-identity class is an element of LD-IdClasses. Let us consider  $\varphi$ . The functor LD-IdClassOf  $\varphi$  yielding an LD-identity class is defined by the term

(Def. 41)  $[\varphi]_{\text{LD-IdR}}$ .

Now we state the proposition:

$$(32) \quad \varphi =_{\text{LD}} \psi \text{ if and only if } \text{LD-IdClassOf } \varphi = \text{LD-IdClassOf } \psi.$$

PROOF: If  $\varphi =_{\text{LD}} \psi$ , then  $\text{LD-IdClassOf } \varphi = \text{LD-IdClassOf } \psi$  by [14, (18), (23)].  $\square$

The scheme *UnOpCongr* deals with a non empty set  $\mathcal{X}$  and a unary functor  $\mathcal{F}$  yielding an element of  $\mathcal{X}$  and an equivalence relation  $\mathcal{E}$  of  $\mathcal{X}$  and states that

(Sch. 2) There exists a unary operation  $f$  on Classes  $\mathcal{E}$  such that for every element

$$x \text{ of } \mathcal{X}, f([x]_{\mathcal{E}}) = [\mathcal{F}(x)]_{\mathcal{E}}$$

provided

- for every elements  $x, y$  of  $\mathcal{X}$  such that  $\langle x, y \rangle \in \mathcal{E}$  holds  $\langle \mathcal{F}(x), \mathcal{F}(y) \rangle \in \mathcal{E}$ .

The scheme *BinOpCongr* deals with a non empty set  $\mathcal{X}$  and a binary functor  $\mathcal{F}$  yielding an element of  $\mathcal{X}$  and an equivalence relation  $\mathcal{E}$  of  $\mathcal{X}$  and states that



(Sch. 3) There exists a binary operation  $f$  on Classes  $\mathcal{E}$  such that for every elements  $x, y$  of  $\mathcal{X}$ ,  $f([x]_{\mathcal{E}}, [y]_{\mathcal{E}}) = [\mathcal{F}(x, y)]_{\mathcal{E}}$

provided

- for every elements  $x_1, x_2, y_1, y_2$  of  $\mathcal{X}$  such that  $\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \in \mathcal{E}$  holds  $\langle \mathcal{F}(x_1, y_1), \mathcal{F}(x_2, y_2) \rangle \in \mathcal{E}$ .

From now on  $x, y, z$  denote LD-identity classes.

Now we state the proposition:

(33) There exists  $\varphi$  such that  $x = \text{LD-IdClassOf } \varphi$ .

Let us consider  $x$ . We say that  $x$  is LD-provable if and only if

(Def. 42) there exists  $\varphi$  such that  $x = \text{LD-IdClassOf } \varphi$  and  $\varphi$  is LD-provable.

The functor  $\neg x$  yielding an LD-identity class is defined by

(Def. 43) there exists  $\varphi$  such that  $x = \text{LD-IdClassOf } \varphi$  and  $it = \text{LD-IdClassOf } \neg\varphi$ .

One can verify that the functor is involutive. Let us consider  $y$ . The functor  $x \wedge y$  yielding an LD-identity class is defined by

(Def. 44) there exists  $\varphi$  and there exists  $\psi$  such that  $x = \text{LD-IdClassOf } \varphi$  and  $y = \text{LD-IdClassOf } \psi$  and  $it = \text{LD-IdClassOf } (\varphi \wedge \psi)$ .

Note that the functor is commutative and idempotent. The functor  $x=y$  yielding an LD-identity class is defined by

(Def. 45) there exists  $\varphi$  and there exists  $\psi$  such that  $x = \text{LD-IdClassOf } \varphi$  and  $y = \text{LD-IdClassOf } \psi$  and  $it = \text{LD-IdClassOf } \varphi=\psi$ .

One can check that the functor is commutative.

The functor  $x \vee y$  yielding an LD-identity class is defined by the term

(Def. 46)  $\neg(\neg x \wedge \neg y)$ .

Let us observe that the functor is commutative and idempotent. The functor  $x \Rightarrow y$  yielding an LD-identity class is defined by the term

(Def. 47)  $x=(x \wedge y)$ .

Let  $\varphi$  be an LD-provable GRZ-formula. Let us observe that  $\text{LD-IdClassOf } \varphi$  is LD-provable.

Now we state the proposition:

(34) If  $\text{LD-IdClassOf } \varphi$  is LD-provable, then  $\varphi$  is LD-provable. The theorem is a consequence of (32) and (21).

Let us consider  $x$  and  $y$ . Now we state the propositions:

(35)  $x \wedge y$  is LD-provable if and only if  $x$  is LD-provable and  $y$  is LD-provable. The theorem is a consequence of (34) and (20).

(36)  $x=y$  is LD-provable if and only if  $x = y$ . The theorem is a consequence of (34) and (32).

Now we state the proposition:

$$(37) \quad \text{LD-IdClassOf } \neg\varphi = \neg \text{LD-IdClassOf } \varphi.$$

Let us consider  $\varphi$  and  $\psi$ . Now we state the propositions:

$$(38) \quad \text{LD-IdClassOf}(\varphi \wedge \psi) = \text{LD-IdClassOf } \varphi \wedge \text{LD-IdClassOf } \psi.$$

$$(39) \quad \text{LD-IdClassOf } \varphi = \psi = (\text{LD-IdClassOf } \varphi) = (\text{LD-IdClassOf } \psi).$$

$$(40) \quad \text{LD-IdClassOf}(\varphi \vee \psi) = \text{LD-IdClassOf } \varphi \vee \text{LD-IdClassOf } \psi.$$

$$(41) \quad \text{LD-IdClassOf}(\varphi \Rightarrow \psi) = \text{LD-IdClassOf } \varphi \Rightarrow \text{LD-IdClassOf } \psi.$$

Now we state the propositions:

$$(42) \quad (x \wedge y) \wedge z = x \wedge (y \wedge z). \text{ The theorem is a consequence of (33) and (32).}$$

$$(43) \quad x \Rightarrow y \text{ is LD-provable if and only if } x = x \wedge y.$$

$$(44) \quad \text{If } x \Rightarrow y \text{ is LD-provable and } y \Rightarrow z \text{ is LD-provable, then } x \Rightarrow z \text{ is LD-provable. The theorem is a consequence of (36) and (42).}$$

$$(45) \quad \text{If } \varphi \Rightarrow \psi \text{ is LD-provable and } \psi \Rightarrow \vartheta \text{ is LD-provable, then } \varphi \Rightarrow \vartheta \text{ is LD-provable. The theorem is a consequence of (41), (34), and (44).}$$

Let us consider  $x$ ,  $y$ , and  $z$ . Now we state the propositions:

$$(46) \quad x \vee (y \vee z) = (x \vee y) \vee z.$$

$$(47) \quad x \wedge (y \vee z) = x \wedge y \vee x \wedge z. \text{ The theorem is a consequence of (33), (32), and (40).}$$

$$(48) \quad x \vee y \wedge z = (x \vee y) \wedge (x \vee z). \text{ The theorem is a consequence of (47).}$$

Let us consider  $x$  and  $y$ . Now we state the propositions:

$$(49) \quad x \Rightarrow y \text{ is LD-provable and } y \Rightarrow x \text{ is LD-provable if and only if } x = y. \text{ The theorem is a consequence of (36).}$$

$$(50) \quad \text{If } x \text{ is LD-provable, then } x \vee y \text{ is LD-provable. The theorem is a consequence of (33), (35), (47), and (48).}$$

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