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Introduction to Liouville Numbers

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Summary. The article defines Liouville numbers, originally introduced by Joseph Liouville in 1844 [17] as an example of an object which can be approximated "quite closely" by a sequence of rational numbers. A real number x is a Liouville number iff for every positive integer n, there exist integers p and q such that $q > 1$ and

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

It is easy to show that all Liouville numbers are irrational. Liouville constant, which is also defined formally, is the first transcendental (not algebraic) number. It is defined in Section 6 quite generally as the sum

$$\sum_{k=1}^{\infty} \frac{a_k}{b^{k!}}$$

for a finite sequence $\{a_k\}_{k \in \mathbb{N}}$ and $b \in \mathbb{N}$. Based on this definition, we also introduced the so-called Liouville number as

$$L = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001\ldots,$$

substituting in the definition of $L(a_k, b)$ the constant sequence of 1's and $b = 10$. Another important examples of transcendental numbers are e and π [7], [13], [6]. At the end, we show that the construction of an arbitrary Lioville constant satisfies the properties of a Liouville number [12], [1]. We show additionally, that the set of all Liouville numbers is infinite, opening the next item from Abad and Abad's list of "Top 100 Theorems". We show also some preliminary constructions linking real sequences and finite sequences, where summing formulas are involved. In the Mizar [14] proof, we follow closely `https://en.wikipedia.org/wiki/Liouville_number`. The aim is to show that all Liouville numbers are transcendental.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers x, y. If $x > 1$ and $y > 1$, then $x \cdot y \geqslant x + y$.

Let us consider a natural number n. Now we state the propositions:

(2) $n \leqslant n!$.

(3) $n \cdot n! = (n+1)! - n!$.

(4) If $n \geqslant 1$, then $2 \leqslant (n+1)!$.

Let us consider natural numbers n, i. Now we state the propositions:

(5) If $n \geqslant 1$ and $i \geqslant 1$, then $(n+i)! \geqslant n! + i$.

(6) If $n \geqslant 2$ and $i \geqslant 1$, then $(n+i)! > n! + i$. The theorem is a consequence of (1).

(7) Let us consider a natural number b. If $b > 1$, then $|\frac{1}{b}| < 1$.

(8) Let us consider an integer d. Then there exists a non zero natural number n such that $2^{n-1} > d$.

Let a be an integer and b be a natural number. Note that a^b is integer.

2. Sequences

Now we state the propositions:

(9) Let us consider sequences s_1, s_2 of real numbers. Suppose for every natural number n, $0 \leqslant s_1(n) \leqslant s_2(n)$ and there exists a natural number n such that $1 \leqslant n$ and $s_1(n) < s_2(n)$ and s_2 is summable. Then

  (i) s_1 is summable, and

  (ii) $\sum s_1 < \sum s_2$.

(10) Let us consider a sequence f of real numbers. Suppose there exists a natural number n such that for every natural number k such that $k \geqslant n$ holds $f(k) = 0$. Then f is summable.

PROOF: Set $p = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$. Reconsider $p_2 = p(n)$ as a real number. Set $r = \{p_2\}_{n \in \mathbb{N}}$. For every natural number k such that $k \geqslant n$ holds $p(k) = r(k)$ by [15, (57)], [3, (12)]. $\square$

(11) Let us consider a natural number b. If $b > 1$, then $\sum((\frac{1}{b})^\kappa)_{\kappa\in\mathbb{N}} = \frac{b}{b-1}$. The theorem is a consequence of (7).

Let n be a natural number. Let us observe that $\{n\}_{n\in\mathbb{N}}$ is $\mathbb{N}$-valued.

Let r be a positive natural number. Note that $\{r\}_{n\in\mathbb{N}}$ is positive yielding and there exists a sequence of real numbers which is $\mathbb{N}$-valued and $\mathbb{Z}$-valued.

Now we state the propositions:

(12) Let us consider a sequence F of real numbers, a natural number n, and a real number a. Suppose for every natural number k, $F(k) = a$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(n) = a \cdot (n + 1)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = a \cdot (\$_1 + 1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i, $\mathcal{P}[i]$ from [3, Sch. 2]. □

(13) Let us consider a natural number n, and a real number a. Then $(\sum_{\alpha=0}^{\kappa} (\{a\}_{n\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = a \cdot (n + 1)$. The theorem is a consequence of (12).

Let f be a $\mathbb{Z}$-valued sequence of real numbers. Note that $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$ is $\mathbb{Z}$-valued.

Let f be an $\mathbb{N}$-valued sequence of real numbers. Observe that $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$ is $\mathbb{N}$-valued.

Now we state the propositions:

(14) Let us consider a sequence f of real numbers. Suppose there exists a natural number n such that for every natural number k such that $k \geqslant n$ holds $f(k) = 0$. Then there exists a natural number n such that for every natural number k such that $k \geqslant n$ holds $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n)$.

PROOF: Set $p = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$. Reconsider $p_2 = p(n)$ as a real number. Set $r = \{p_2\}_{n\in\mathbb{N}}$. For every natural number k such that $k \geqslant n$ holds $p(k) = r(k)$ by [15, (57)], [3, (12)]. □

(15) Let us consider a $\mathbb{Z}$-valued sequence f of real numbers. Suppose there exists a natural number n such that for every natural number k such that $k \geqslant n$ holds $f(k) = 0$. Then $\sum f$ is an integer.

PROOF: Set $p = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$. Reconsider $p_2 = p(n)$ as a real number. Set $r = \{p_2\}_{n\in\mathbb{N}}$. For every natural number k such that $k \geqslant n$ holds $p(k) = r(k)$ by [15, (57)], [3, (12)]. □

Let f be a non-negative yielding sequence of real numbers and n be a natural number.

One can verify that $f \uparrow n$ is non-negative yielding.

3. Transformations between Real Functions and Finite Sequences

Let f be a sequence of real numbers and X be a subset of $\mathbb{N}$. The functor $f|X$ yielding a sequence of real numbers is defined by the term

(Def. 1) $(\mathbb{N} \longmapsto 0)+\cdot f\restriction X$.

Note that $f\restriction X$ is $\mathbb{N}$-defined.

Let n be a natural number. Let us note that $f|\operatorname{Seg} n$ is summable.

Let f be a $\mathbb{Z}$-valued sequence of real numbers. One can verify that $f|\operatorname{Seg} n$ is $\mathbb{Z}$-valued.

Now we state the proposition:

(16) Let us consider a sequence f of real numbers. Then $f|\operatorname{Seg} 0 = \{0\}_{n\in\mathbb{N}}$.
Proof: Set $f_3 = f|\operatorname{Seg} 0$. Set $g = \{0\}_{n\in\mathbb{N}}$. For every element x of $\mathbb{N}$, $f_3(x) = g(x)$ by [10, (11)]. $\square$

Let f be a sequence of real numbers and n be a natural number. The functor $\operatorname{FinSeq}(f, n)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by the term

(Def. 2) $f\restriction\operatorname{Seg} n$.

Now we state the proposition:

(17) Let us consider a sequence f of real numbers, and natural numbers k, n. If $k \in \operatorname{Seg} n$, then $(f|\operatorname{Seg} n)(k) = f(k)$.

Let us consider a sequence f of real numbers and a natural number n. Now we state the propositions:

(18) If $f(0) = 0$, then $\sum \operatorname{FinSeq}(f, n) = \sum(f|\operatorname{Seg} n)$.
Proof: Set $f_1 = f|\operatorname{Seg} n$. Set $g = \operatorname{FinSeq}(f, n)$. Reconsider $f_0 = f(0)$ as an element of $\mathbb{R}$. Set $h = \langle f_0\rangle \frown g$. For every natural number k such that $k < n+1$ holds $f_1(k) = h(k+1)$ by [3, (13), (14)], [22, (25)], [8, (49)]. For every natural number k such that $k \geqslant n+1$ holds $f_1(k) = 0$ by [3, (16)], [4, (1)], [24, (57)], [10, (11)]. $\square$

(19) $\operatorname{dom}\operatorname{FinSeq}(f, n) = \operatorname{Seg} n$.

(20) Let us consider a sequence f of real numbers, and a natural number i. Then $\operatorname{FinSeq}(f, i) \frown \langle f(i+1)\rangle = \operatorname{FinSeq}(f, i+1)$.
Proof: Set $f_1 = \operatorname{FinSeq}(f, i)$. Set $g = \langle f(i+1)\rangle$. Set $h = \operatorname{FinSeq}(f, i+1)$. $\operatorname{dom} f_1 = \operatorname{Seg} i$. For every natural number k such that $k \in \operatorname{dom}(f_1 \frown g)$ holds $(f_1 \frown g)(k) = h(k)$ by [3, (13)], [4, (5), (25)], (19). $\square$

Let us consider a sequence f of real numbers and a natural number n. Now we state the propositions:

(21) If $f(0) = 0$, then $\sum \operatorname{FinSeq}(f, n) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv \sum \operatorname{FinSeq}(f, \$_1) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}$

($\$_1$). For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by (20), [23, (4)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. $\square$

(22) If $f(0) = 0$, then $\sum(f|\operatorname{Seg} n) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa\in\mathbb{N}}(n)$. The theorem is a consequence of (21) and (18).

(23) Let us consider a $\mathbb{Z}$-valued sequence f of real numbers, and a natural number n. If $f(0) = 0$, then $\sum(f|\operatorname{Seg} n)$ is an integer. The theorem is a consequence of (22).

(24) Let us consider a sequence f of real numbers, and a natural number n. Suppose f is summable and $f(0) = 0$. Then $\sum f = \sum \operatorname{FinSeq}(f, n) + \sum(f\uparrow(n+1))$. The theorem is a consequence of (21).

One can check that there exists a sequence of real numbers which is positive yielding and $\mathbb{N}$-valued.

4. Sequences not Vanishing at Infinity

Let f be a sequence of real numbers. We say that f is eventually non-zero if and only if

(Def. 3) for every natural number n, there exists a natural number N such that $n \leqslant N$ and $f(N) \neq 0$.

Observe that every sequence of real numbers which is eventually nonzero is also eventually non-zero and $\mathrm{id}_{\mathrm{seq}}(\mathrm{id}_{\mathbb{N}})$ is eventually nonzero and there exists a sequence of real numbers which is eventually non-zero.

Now we state the proposition:

(25) Let us consider an eventually non-zero sequence f of real numbers, and a natural number n. Then $f\uparrow n$ is eventually non-zero.

Let f be an eventually non-zero sequence of real numbers and n be a natural number. Note that $f\uparrow n$ is eventually non-zero as a sequence of real numbers and every sequence of real numbers which is non-zero and constant is also eventually non-zero.

Let b be a natural number. The functor $\operatorname{pfact}(b)$ yielding a sequence of real numbers is defined by

(Def. 4) for every natural number i, $it(i) = \frac{1}{b^{i!}}$.

Now we state the propositions:

(26) Let us consider natural numbers b, i. Suppose $b \geqslant 1$. Then $(\operatorname{pfact}(b))(i) \leqslant ((\frac{1}{b})^{\kappa})_{\kappa\in\mathbb{N}}(i)$.

(27) Let us consider a natural number b. Suppose $b > 1$. Then

(i) $\operatorname{pfact}(b)$ is summable, and

(ii) $\sum \mathrm{pfact}(b) \leqslant \frac{b}{b-1}$.

The theorem is a consequence of (26) and (11).

Let b be a non trivial natural number. Observe that $\mathrm{pfact}(b)$ is summable and there exists a sequence of real numbers which is non-negative yielding.

Now we state the proposition:

(28) Let us consider natural numbers n, b. Suppose $b > 1$ and $n \geqslant 1$. Then $\sum((b-1) \cdot (\mathrm{pfact}(b) \uparrow (n+1))) < \frac{1}{(b^{n!})^n}$.

PROOF: $\mathrm{pfact}(b) \uparrow (n+1)$ is summable. Set $s_1 = \mathrm{pfact}(b) \uparrow (n+1)$. Set $s_2 = ((\frac{1}{b})^\kappa)_{\kappa \in \mathbb{N}} \uparrow (n+1)!$. For every natural number k, $0 \leqslant s_1(k) \leqslant s_2(k)$ by [3, (13)], [19, (7)], [3, (16)], [5, (8)]. There exists a natural number k such that $1 \leqslant k$ and $s_1(k) < s_2(k)$ by [19, (7)], [20, (39)]. $\sum s_1 < \sum s_2$. Reconsider $b_3 = b^{(n+1)!}$ as a natural number. $((\frac{1}{b})^\kappa)_{\kappa \in \mathbb{N}} \uparrow (n+1)! = (\frac{1}{b_3}) \cdot ((\frac{1}{b})^\kappa)_{\kappa \in \mathbb{N}}$ by [16, (8)], [19, (7)], [9, (63)]. $\square$

5. Liouville Numbers

Let x be a real number. We say that x is Liouville if and only if

(Def. 5) for every natural number n, there exists an integer p and there exists a natural number q such that $q > 1$ and $0 < |x - \frac{p}{q}| < \frac{1}{q^n}$.

Now we state the proposition:

(29) Let us consider a real number r. Then r is Liouville if and only if for every non zero natural number n, there exists an integer p and there exists a natural number q such that $1 < q$ and $0 < |r - \frac{p}{q}| < \frac{1}{q^n}$.

Let a be a sequence of real numbers and b be a natural number. The functor $\mathrm{LiouvilleSeq}(a, b)$ yielding a sequence of real numbers is defined by

(Def. 6) $it(0) = 0$ and for every non zero natural number k, $it(k) = \frac{a(k)}{b^{k!}}$.

One can check that every real number which is Liouville is also irrational.

6. Liouville Constant

Let a be a sequence of real numbers and b be a natural number. The functor $\mathrm{LiouvilleConst}(a, b)$ yielding a real number is defined by the term

(Def. 7) $\sum \mathrm{LiouvilleSeq}(a, b)$.

The functor $\mathrm{BLiouvilleSeq}(b)$ yielding a sequence of real numbers is defined by

(Def. 8) for every natural number n, $it(n) = b^{n!}$.

Let us note that BLiouvilleSeq(b) is $\mathbb{N}$-valued.

Let a be a sequence of real numbers. The functor ALiouvilleSeq(a, b) yielding a sequence of real numbers is defined by

(Def. 9) for every natural number n, $it(n) =$ $(\text{BLiouvilleSeq}(b))(n) \cdot \sum(\text{LiouvilleSeq}(a, b) | \operatorname{Seg} n)$.

Now we state the propositions:

(30) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and natural numbers b, n, k. Suppose $b > 0$ and $k \leqslant n$. Then $(\text{LiouvilleSeq}(a, b))(k) \cdot (\text{BLiouvilleSeq}(b))(n)$ is an integer.

(31) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and natural numbers b, n. If $b > 0$, then $(\text{ALiouvilleSeq}(a, b))(n)$ is an integer.

Proof: Set $L = \text{LiouvilleSeq}(a, b)$. Set $B = \text{BLiouvilleSeq}(b)$. Set $f_3 = B(n) \cdot (L | \operatorname{Seg} n)$. $\operatorname{rng} f_3 \subseteq \mathbb{Z}$ by [4, (1)], [24, (62)], [10, (13)], [8, (49)]. Set $m = n + 1$. For every natural number k such that $k \geqslant m$ holds $f_3(k) = 0$ by [3, (13)], [4, (1)], [24, (57)], [10, (11)]. □

Let a be an $\mathbb{N}$-valued sequence of real numbers and b be a non zero natural number. Let us observe that ALiouvilleSeq(a, b) is $\mathbb{Z}$-valued.

Now we state the propositions:

(32) Let us consider non zero natural numbers n, b. If $b > 1$, then $(\text{BLiouvilleSeq}(b))(n) > 1$.

(33) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and a non zero natural number b. Suppose $b \geqslant 2$ and $\operatorname{rng} a \subseteq b$. Then LiouvilleSeq($a, b$) is summable.

Proof: Set $f = \text{LiouvilleSeq}(a, b)$. For every natural number i, $\frac{b-1}{b^{i!}} = ((b-1) \cdot \text{pfact}(b))(i)$. For every natural number i, $f(i) \geqslant 0$ and $f(i) \leqslant ((b-1) \cdot \text{pfact}(b))(i)$ by [21, (3)], [16, (12)], [3, (51), (44), (13)]. pfact(b) is summable. □

(34) Let us consider a sequence a of real numbers, a non zero natural number n, and a non zero natural number b. Suppose $b > 1$. Then $\frac{(\text{ALiouvilleSeq}(a,b))(n)}{(\text{BLiouvilleSeq}(b))(n)} = \sum \text{FinSeq}(\text{LiouvilleSeq}(a, b), n)$. The theorem is a consequence of (32) and (18).

(35) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, a non trivial natural number b, and a natural number n. Then $(\text{LiouvilleSeq}(a, b))(n) \geqslant 0$.

(36) Let us consider a positive yielding, $\mathbb{N}$-valued sequence a of real numbers, a non trivial natural number b, and a non zero natural number n. Then $(\text{LiouvilleSeq}(a, b))(n) > 0$.

Let a be an $\mathbb{N}$-valued sequence of real numbers and b be a non trivial natural number. One can check that LiouvilleSeq(a, b) is non-negative yielding.

Now we state the propositions:

(37) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and natural numbers b, c. Suppose $b \geqslant 2$ and $c \geqslant 1$ and $\operatorname{rng} a \subseteq c$ and $c \leqslant b$. Let us consider a natural number i. Then $(\text{LiouvilleSeq}(a, b))(i) \leqslant ((c-1) \cdot \text{pfact}(b))(i)$.

(38) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and natural numbers b, c. Suppose $b \geqslant 2$ and $c \geqslant 1$ and $\operatorname{rng} a \subseteq c$ and $c \leqslant b$. Then $\sum \text{LiouvilleSeq}(a, b) \leqslant \sum((c-1) \cdot \text{pfact}(b))$. The theorem is a consequence of (27), (35), and (37).

(39) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and natural numbers b, c, n. Suppose $b \geqslant 2$ and $c \geqslant 1$ and $\operatorname{rng} a \subseteq c$ and $c \leqslant b$. Then $\sum(\text{LiouvilleSeq}(a, b) \upharpoonright (n+1)) \leqslant \sum((c-1) \cdot (\text{pfact}(b) \upharpoonright (n+1)))$.

PROOF: Set $g = (c-1) \cdot (\text{pfact}(b) \upharpoonright (n+1))$. $\text{pfact}(b) \upharpoonright (n+1)$ is summable. Set $f = \text{LiouvilleSeq}(a, b) \upharpoonright (n+1)$. For every natural number i, $0 \leqslant f(i)$ by [8, (3)]. For every natural number i, $f(i) \leqslant g(i)$ by [15, (9)], (37). $\square$

(40) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, a non trivial natural number b, and a natural number n. Suppose a is eventually non-zero and $\operatorname{rng} a \subseteq b$. Then $\sum(\text{LiouvilleSeq}(a, b) \upharpoonright (n+1)) > 0$.

PROOF: Set $L = \text{LiouvilleSeq}(a, b) \upharpoonright (n+1)$. For every natural number i, $0 \leqslant L(i)$. There exists a natural number i such that $i \in \operatorname{dom} L$ and $0 < L(i)$ by [21, (5)]. Consider k being a natural number such that $k \in \operatorname{dom} L$ and $L(k) > 0$. LiouvilleSeq(a, b) is summable. $\square$

(41) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and a non trivial natural number b. Suppose $\operatorname{rng} a \subseteq b$ and a is eventually non-zero. Let us consider a non zero natural number n. Then there exists an integer p and there exists a natural number q such that $q > 1$ and $0 < |\,\text{LiouvilleConst}(a, b) - \frac{p}{q}| < \frac{1}{q^n}$. The theorem is a consequence of (32), (33), (40), (24), (34), (39), and (28).

The functor LiouvilleConst yielding a real number is defined by the term

(Def. 10) $\text{LiouvilleConst}(\{1\}_{n \in \mathbb{N}}, 10)$.

Now we state the proposition:

(42) Let us consider an $\mathbb{N}$-valued sequence a of real numbers, and a non trivial natural number b. Suppose $\operatorname{rng} a \subseteq b$ and a is eventually non-zero. Then LiouvilleConst(a, b) is Liouville. The theorem is a consequence of (41) and (29).

One can check that LiouvilleConst is Liouville and there exists a real number which is Liouville.

A Liouville number is a Liouville real number. Now we state the propositions:

(43) Let us consider non zero natural numbers m, n. Then $(\mathrm{LiouvilleSeq}(\{1\}_{n\in\mathbb{N}}, m))(n) = m^{-n!}$.

(44) Let us consider a natural number m. If $1 < m$, then $\mathrm{LiouvilleSeq}(\{1\}_{n\in\mathbb{N}}, m)$ is negligible.

Proof: There exists a function f from $\mathbb{N}$ into $\mathbb{R}$ such that for every natural number x, $f(x) = \frac{1}{2^x}$. Consider f being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every natural number x, $f(x) = \frac{1}{2^x}$. Set $g = \mathrm{LiouvilleSeq}(\{1\}_{n\in\mathbb{N}}, m)$. For every natural number x, $|g(x)| \leqslant |f(x)|$ by [18, (5), (4)]. $\square$

(45) $\frac{1}{10} < \mathrm{LiouvilleConst} \leqslant \frac{10}{9} - \frac{1}{10}$.

Proof: Set $a = \{1\}_{n\in\mathbb{N}}$. Set $b = 10$. Reconsider $n = 1$ as a non zero natural number. Set $f = \mathrm{LiouvilleSeq}(a, b)$. Set $p_1 = \mathrm{pfact}(b)$. f is summable. For every natural number n, $0 \leqslant f(n)$. $f(1) = 10^{-1}$. Set $s_1 = f \uparrow 2$. Set $s_2 = p_1 \uparrow 2$. $\sum p_1 = (\sum_{\alpha=0}^{\kappa} p_1(\alpha))_{\kappa\in\mathbb{N}}(1) + \sum(p_1 \uparrow (1+1))$. $\sum p_1 \leqslant \frac{b}{b-1}$. s_2 is summable. For every natural number n, $0 \leqslant s_1(n) \leqslant s_2(n)$ by (37), [11, (7)], [2, (50)], (35). $\square$

(46) Let us consider a Liouville number n_1, and an integer z. Then $z + n_1$ is Liouville. The theorem is a consequence of (29).

Let n_1 be a Liouville number and z be an integer. One can verify that $n_1 + z$ is Liouville.

The set of all Liouville numbers yielding a subset of $\mathbb{R}$ is defined by the term

(Def. 11) the set of all n_1 where n_1 is a Liouville number.

Note that the set of all Liouville numbers is infinite.

References

[1] Tom M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Springer-Verlag, 2nd edition, 1997.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Grzegorz Bancerek and Piotr Rudnicki. Two programs for **SCM**. Part I – preliminaries. *Formalized Mathematics*, 4(**1**):69–72, 1993.

[6] Sophie Bernard, Yves Bertot, Laurence Rideau, and Pierre-Yves Strub. Formal proofs of transcendence for e and π as an application of multivariate and symmetric polynomials. In Jeremy Avigad and Adam Chlipala, editors, *Proceedings of the 5th ACM SIGPLAN Conference on Certified Programs and Proofs*, pages 76–87. ACM, 2016. doi:10.1145/2854065.2854072.

[7] Jesse Bingham. Formalizing a proof that e is transcendental. *Journal of Formalized Reasoning*, 4:71–84, 2011.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[11] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[12] J.H. Conway and R.K. Guy. *The Book of Numbers*. Springer-Verlag, 1996.

[13] Manuel Eberl. Liouville numbers. *Archive of Formal Proofs*, December 2015. `http://isa-afp.org/entries/Liouville_Numbers.shtml`, Formal proof development.

[14] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[15] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[16] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[17] Joseph Liouville. Nouvelle démonstration d'un théorème sur les irrationnelles algébriques, inséré dans le Compte Rendu de la dernière séance. *Compte Rendu Acad. Sci. Paris*, Sér.A (18):910–911, 1844.

[18] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[19] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[20] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[21] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[22] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[23] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1 (**5**):979–981, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

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