

On the weak non-defectivity of Veronese embeddings of projective spaces*

Edoardo Ballico[†]

*Department of Mathematics,
University of Trento,
38050 Povo (TN), Italy*

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Abstract: Fix integers n, x, k such that $n \geq 3$, $k > 0$, $x \geq 4$, $(n, x) \neq (3, 4)$ and $k(n+1) < \binom{n+x}{n}$. Here we prove that the order x Veronese embedding of \mathbf{P}^n is not weakly $(k-1)$ -defective, i.e. for a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = k+1$ the projective space $|\mathcal{I}_{2S}(x)|$ of all degree t hypersurfaces of \mathbf{P}^n singular at each point of S has dimension $\binom{n+x}{n} - 1 - k(n+1)$ (proved by Alexander and Hirschowitz) and a general $F \in |\mathcal{I}_{2S}(x)|$ has an ordinary double point at each $P \in S$ and $\text{Sing}(F) = S$.

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1 Introduction

The main aim of this paper is to use the so-called Horace Method introduced by A. Hirschowitz to prove the following result.

Theorem 1.1. *Fix integers $n \geq 3$, $x \geq 4$, $(n, x) \neq (3, 4)$ and $k \geq 0$ such that*

$$k(n+1) < \binom{n+x}{n} \tag{1}$$

and a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = k$. Let $|\mathcal{I}_{2S}(x)|$ denote the projective space of all degree x hypersurfaces of \mathbf{P}^n singular at each point of S . Then $\dim(|\mathcal{I}_{2S}(t)|) =$

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[†] E-mail: ballico@science.unitn.it

$\binom{n+x}{n} - k(n+1) - 1$. A general $F \in |\mathcal{I}_{2S}(x)|$ satisfies $\text{Sing}(F) = S$ and it has an ordinary node at each point of S .

The computation of $\dim(|\mathcal{I}_{2S}(x)|)$ is an important theorem due to J. Alexander and A. Hirschowitz [1, 2, 3, 4, 6]. With the classical terminology their theorem means that the degree x Veronese embedding of \mathbf{P}^n is not $(k-1)$ -defective. They also give the list of all triples (n, x, k) such that the degree $x \geq 3$ Veronese embedding of \mathbf{P}^n is $(k-1)$ -defective: the triples $(n, x, k) \in \{(2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)\}$ ([6], Th. 1). The new (I hope) part is that $\text{Sing}(F) = S$ and that F has an ordinary node at each point of S for a general $F \in |\mathcal{I}_{2S}(x)|$. For each fixed pair (n, x) there is at most one integer k ($k = [(\binom{n+x}{n} - 1)/(n+1)]$) satisfying (1) and for which Theorem 1.1 is not known to be true by [10], Cor. 4.5. As obvious to everybody working on this topic the case $k = [(\binom{n+x}{n} - 1)/(n+1)]$ is by far the most difficult. To prove it we will use an idea due to M. Mella and used in [10]. For related examples in which the singular locus has positive dimension, see [9] and [8], Remark 6.2, which quotes [11], and [10], Remark 4.4.

We work over an algebraically closed field \mathbb{K} with $\text{char}(\mathbb{K}) = 0$. Our proof of Theorem 1.1 heavily depends on the characteristic zero assumption: a key tool will be [7], Th. 1.4. We borrowed a key idea from [10].

2 The proof

For any scheme A and any $P \in A_{\text{reg}}$ let $2P$ (or $2\{P, A\}$ if there is any danger of misunderstanding) denote the first infinitesimal neighborhood of P in A , i.e. the closed subscheme of A with $(\mathcal{I}_P)^2$ as its ideal sheaf. Hence $2P$ is a zero-dimensional subscheme of A and $\text{length}(2P) = \dim(A) + 1$. For any finite subset $S \subset A_{\text{reg}}$, set $2S := \cup_{P \in S} 2P$ and $2\{S, A\} := \cup_{P \in S} 2\{P, A\}$. For any closed subscheme Z of A and every effective Cartier divisor D of A let $\text{Res}_D(Z)$ denote the residual scheme of Z with respect to D , i.e. the closed subscheme of A with $\mathcal{I}_{Z,A} : \mathcal{I}_{D,A}$ as its ideal sheaf. For any effective Cartier divisor D of A such that $P \in D_{\text{reg}}$ we have $2\{P, A\} \cap D = 2\{P, D\}$ and $\text{Res}_D(2\{P, A\}) = \{P\}$.

We will often use the following elementary form of the so-called Horace Lemma.

Lemma 2.1. *Let $H \subset \mathbf{P}^n$ be a hyperplane and $Z \subset \mathbf{P}^n$ a closed subscheme. Then:*

- (a) $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^0(\mathbf{P}^n, \mathcal{I}_{\text{Res}_H(Z)}(d-1)) + h^0(H, \mathcal{I}_{Z \cap H}(d));$
- (b) $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^1(\mathbf{P}^n, \mathcal{I}_{\text{Res}_H(Z)}(d-1)) + h^1(H, \mathcal{I}_{Z \cap H}(d)).$

Proof. By the very definition of a residual scheme with respect to H , there is the following exact sequence

$$0 \rightarrow \mathcal{I}_{\text{Res}_H(Z)}(d-1) \rightarrow \mathcal{I}_Z(d) \rightarrow \mathcal{I}_{Z \cap H}(d) \rightarrow 0 \quad (2)$$

whose long cohomology exact sequence proves the lemma. \square

The following result is a very particular case of [5], Lemma 2.3 (see in particular Fig. 1 on p. 308).

Lemma 2.2. *Let $H \subset \mathbf{P}^n$ be hyperplane, $Z \subset \mathbf{P}^n$ a closed subscheme not containing H and s a positive integer. Let U be the union of Z and s general double points of*

\mathbf{P}^n . Let S be the union of s general points of H . Let $E \subset H$ be the union of s general double points of H (not double points of \mathbf{P}^n , i.e. each of them has length n). To prove $h^1(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$) it is sufficient to prove $h^1(H, \mathcal{I}_{(Z \cap H) \cup S}(d)) = h^1(\mathbf{P}^n, \mathcal{I}_{Res_H(Z) \cup E}(d-1)) = 0$ (resp. $h^0(H, \mathcal{I}_{(Z \cap H) \cup S}(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{Res_H(Z) \cup E}(d-1)) = 0$).

For all integers $n > 0, t > 0$ define the integers $a_{n,t}, b_{n,t}, c_{n,t}$ and $d_{n,t}$ using the following relations:

$$(n + 1)a_{n,t} + b_{n,t} = \binom{n + t}{n}, \quad 0 \leq b_{n,t} \leq n \tag{3}$$

$$(n + 1)c_{n,t} + d_{n,t} + 1 = \binom{n + t}{n}, \quad 0 \leq d_{n,t} \leq n \tag{4}$$

Notice that $c_{n,t} = a_{n,t}$ and $d_{n,t} = b_{n,t} - 1$ if $b_{n,t} > 0$, while $c_{n,t} = a_{n,t} - 1$ and $d_{n,t} = n$ if $b_{n,t} = 0$.

Subtracting (3) from the same equation for the integers $n, t-1$ we obtain the following relation:

$$(n + 1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = \binom{n + t - 1}{n - 1} \tag{5}$$

Using (3) for the integers $n - 1, t$ and (5), we obtain the following relation:

$$(n + 1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = na_{n-1,t} + b_{n-1,t} \tag{6}$$

Lemma 2.3. Fix a hyperplane $H \in \mathbf{P}^n$ and an integer $y > 0$. Let D be an irreducible y -dimensional family of hypersurfaces of \mathbf{P}^n . Then for a general $B \subset H$ such that $\sharp(B) = y$ there is $Y \in D$ such that $B \subset Y$.

Proof. Take a general $P \in H$. Since $y > 0$ there are infinitely many hypersurfaces parametrized by D . Hence the set of all $Y \in D$ containing P is non-empty and contains an irreducible subfamily $D_P \subset D$ of dimension $y - 1$. If $y = 1$, we are done. If $y \geq 2$ we use induction on y and the family D_P for the integer $y' := y - 1$. □

Remark 2.4. If $x \geq 2$, $\sharp(A) = c_{n,x} - a_{n-1,x} - b_{n-1,x}$ and $h^1(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = 0$, then $h^0(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = \binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) = \binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + a_{n-1,x} + (n+1)b_{n-1,x} \binom{n+x-1}{n-1} - b_{n-1,x} = 1 + d_{n,x} + nb_{n-1,x} + a_{n-1,x} \geq 1 + a_{n-1,x} + b_{n-1,x}$.

Lemma 2.5. Assume either $x = 3$ and $n \geq 10$ or $x \geq 4$ and $n \geq 4$. Then

$$(n + 1)(a_{n,x} - a_{n-1,x}) + n + 1 \leq \binom{n + x - 1}{n} \tag{7}$$

Proof. Since $(n + 1)a_{n,x} \leq \binom{n+x}{n}$ and $na_{n-1,x} \geq \binom{n+x-1}{n-1}$, the inequality (7) is satisfied if $2n - 1 \geq a_{n-1,x}$, i.e. if

$$(2n - 1)(n + 1) \leq \binom{n + x - 1}{n - 1} \tag{8}$$

which is satisfied if either $x = 3$ and $n \geq 10$ or $x \geq 4$ and $n \geq 4$. \square

Lemma 2.6. *We have $\binom{n+x-1}{n} \geq c_{n,x}$ for all $n \geq 2$ and $x \geq 3$*

Proof. Since $(n+1)c_{n,x} < \binom{n+x}{n}$, it is sufficient to use the obvious inequality $(n+1)x \geq (n+x)$. \square

Lemma 2.7. *We have $\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq a_{n-1,x}$ for all $n \geq 3$ and $x \geq 3$.*

Proof. By (3) and (4) this inequality is equivalent to the inequality $\binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + \binom{n+x-1}{n-1} + nb_{n-1,x}$, which is obviously satisfied. \square

Proof (of Theorem 1.1). Fix a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = c_{n,x} - a_{n-1,x}$. By Lemma 2.5 and the inequality $c_{n,x} \leq a_{n,x}$ we have $(n+1)\sharp(S) + n + 1 \leq \binom{n+x-1}{n}$. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$ and a general $Y \in |\mathcal{I}_{2S}(x-1)|$ has an ordinary node at each point of S and $\text{Sing}(Y) = S$ ([10], Cor. 4.5). Let $H \subset \mathbf{P}^n$ be a general hyperplane. Hence $H \cap S = \emptyset$ and H is transversal to Y . Now we fix H and S and move Y . Since $n \geq 3$, $Y \cap H$ is irreducible. We fix a general $B \subset H \cap Y$ such that $\sharp(B) = a_{n-1,x}$. \square

Claim 2.8. We have $h^1(\mathbf{P}^n, \mathcal{I}_{2S \cup 2B}(x)) = 0$

Proof (of the Claim). Since $(n-1, x) \notin \{(2, 4), (3, 4), (4, 3), (4, 4)\}$ and B is general in an integral degree $x-1$ hypersurface of H , we have $h^1(H, \mathcal{I}_{2B}(x)) = 0$, i.e. $h^0(H, \mathcal{I}_{2B}(x)) = b_{n-1,x}$. Notice that $\sharp(S) \geq b_{n-1,x}$ (Lemma 2.3). Fix $S'' \subseteq S$ such that $\sharp(S'') = b_{n-1,x}$ and set $S' := S \setminus S''$. S'' may be seen as a set of $b_{n-1,x}$ general points of \mathbf{P}^n . We degenerate it (keeping fixed $S' \cup B$) into a union E of $b_{n-1,x}$ general points of H . Hence $h^0(H, \mathcal{I}_{2B \cup E}(x)) = h^1(H, \mathcal{I}_{2B \cup E}(x)) = 0$. Since $S \cap H = \emptyset$, then $\text{Res}_{2S \cup 2B} = 2S \cup B$. The local deformation space of an ordinary nodal hypersurface singularity is one-dimensional. Since $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$, we obtain that, moving S , the set of the possible nodal hypersurfaces Y is (near Y) at least of dimension $\binom{n+x-1}{n} - 1 - \sharp(S) \geq a_{n-1,x}$ (Lemma 2.6). Hence by Lemma 2.2 to prove the Claim it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\} \cup B}(x-1)) = 0$. First, we will check that $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$. To prove this vanishing it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2E}(x-1)) = 0$. Since $\sharp(E) \leq n$, any n points of \mathbf{P}^n are contained in a hyperplane and S' is chosen independently from H and E , we may consider $S' \cup E$ as a general union of $c_{n,x} - a_{n-1,x}$ points of \mathbf{P}^n . Hence $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2E}(x-1)) = 0$ and thus $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$. By Remark 2.4 and Lemma 2.3 for fixed S' we may take $B \cup E$ general in H . Hence for fixed $S' \cup E$ we may take B general in H . By the generality of B , $S' \cap H = \emptyset$ and $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$, to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\} \cup B}(x-1)) = 0$ and hence the Claim it is sufficient to prove the inequality $h^0(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) \geq \sharp(B)$ (see e.g. [6], Lemma 3), i.e. the

inequality

$$\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) + a_{n-1,x} \quad (9)$$

First assume $c_{n,x} - a_{n-1,x} - b_{n-1,x} \leq a_{n,x-2}$. Then $h^1(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$ and hence $h^0(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E,H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = \binom{n+x-2}{n-1} - nb_{n-1,x}$. Hence the inequality (9) is satisfied in this case. Now assume $c_{n,x} - a_{n-1,x} - b_{n-1,x} > a_{n,x-2}$. Hence $h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$. Hence the inequality (9) is satisfied by Lemma 2.7, proving the Claim. Set $A := Y \cup H$. Since $S \cap H = \emptyset$ and $c_{n,x} > a_{n-1,x}$, at least one of the points of S is an isolated singular point of A . Hence a general $E \in |\mathcal{I}_{2S \cup 2B}(x)|$ has at least one isolated singular point and this point is contained in $S \cup B$, i.e. in a set T such that $h^1(\mathbf{P}^n, \mathcal{I}_{2T}(x)) = 0$ and $\sharp(T) = c_{n,x}$. Hence we may apply the semicontinuity theorem for cohomology and the openness of smoothness to obtain that for a general $G \subset \mathbf{P}^n$ such that $\sharp(G) = c_{n,x}$, the linear system $|\mathcal{I}_{2G}(x)|$ has the expected dimension and a general member of it has an isolated singularity at one point of G . We conclude by [7], Th. 1.4. \square

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