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Subnormal, permutable, and embedded subgroups in finite groups

Research Article

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Abstract: The purpose of this paper is to study the subgroup embedding properties of S-semipermutability, semipermutability, and seminormality. Here we say H is S-semipermutable (resp. semipermutable) in a group G if H permutes which each Sylow subgroup (resp. subgroup) of G whose order is relatively prime to that of H. We say H is seminormal in a group G if H is normalized by subgroups of G whose order is relatively prime to that of H. In particular, we establish that a seminormal p-subgroup is subnormal. We also establish that the solvable groups in which S-permutability is a transitive relation are precisely the groups in which the subnormal subgroups are all S-semipermutable. Local characterizations of this result are also established.

MSC: 20D10, 20D20, 20D35

Keywords: Solvable group • PST-group • Subnormal subgroup • S-semipermutable • Seminormal subgroup

Dedicated to Professor Hermann Heineken
on the occasion of his seventy-fifth birthday.

1. Introduction and statement of results

All groups considered will be finite.

A subgroup H of a group G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be permutable (resp. S-permutable) if it permutes with all the subgroups (resp. Sylow subgroups) of G. Examples of

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permutable subgroups include the normal subgroups of G. However, if G is a modular, non-Dedekind p-group, p a prime, we see permutability is quite different from normality. For instance, letting C_n denote the cyclic group of order n, we see that C_2 is permutable but not normal in the the group $C_8 \rtimes C_2$ where the generator for C_2 maps a generator of C_8 to its fifth power. Kegel [10] proved that an S-permutable subgroup is always subnormal. In particular, as was shown by Ore [12], a permutable subgroup of a group is subnormal. A group G is called a PST-group (resp. PT-group) if S-permutability (resp. permutability) is a transitive relation. By Kegel's result (resp. Ore's result), a group G is a PST-group (resp. PT-group) if every subnormal subgroup of G is S-permutable (resp. permutable) in G. Agrawal [1] (resp. Zachar [16]) classified solvable PST-groups (resp. PT-groups). In [1, Theorem 2], Agrawal showed that a solvable group G is a PST-group if and only if the nilpotent residual of G is an abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms. He also showed that a solvable PST-group G is a PT-group if and only if the Sylow subgroups of G are modular. This is essentially Zachar's characterization of solvable PT-groups. A number of other research papers have been written on these groups. See for example [2–6, 8, 9].

It seems reasonable to investigate other distinguished families of subgroups of a group G which permute with the subnormal subgroups of G. For example in [7], the first author and Heineken studied groups G whose subnormal subgroups permute with all the maximal subgroups of G. These groups are characterized in terms of their Frattini factor group. It was shown by Maier [11] that a subgroup which permutes with all the maximal subgroups of G need not be subnormal. Also, Ballester-Bolinches, Cossey, and Soler-Escrivá [3] considered solvable groups G all of whose subnormal subgroups permute with the system normalizers of G. These groups are closely related to the solvable PST-groups. They also arrive at a nice result which says any subgroup permuting with all system normalizers of a solvable group is necessarily subnormal. It is our purpose here to study subgroup embedding properties different from those mentioned above and to characterize the finite solvable groups whose subnormal subgroups satisfy these different embedding properties.

A subgroup X of a group G is said to be *semipermutable* provided it permutes with every subgroup H of G such that $(|X|,|H|) = 1$. Moreover, X is said to be *S-semipermutable* if it permutes with every Sylow subgroup Y of G such that $(|X|,|Y|) = 1$. An S-semipermutable subgroup of a group need not be subnormal (see Example 4.1). S-semipermutable subgroups have been studied in [2, 15, 18]. A subgroup X of a group G is said to be *seminormal*[1] provided that it is normalized by every subgroup H such that $(|X|,|H|) = 1$. We say X is S-seminormal if it is normalized by every Sylow subgroup Y of G such that $(|X|,|Y|) = 1$.

For the purposes of this article, we will call a group G an SPS-group (resp. SP-group) if all the subnormal subgroups of G are S-semipermutable (resp. semipermutable) in G. We will also call G an *SN-group* provided that every subnormal subgroup of G is seminormal.

We arrive at the following results.

Theorem 1.1.
Let G be a solvable group. Then G is a PST-group if and only if it is an SPS-group.

Theorem 1.2.
A subgroup X of a group G is seminormal if and only if it is S-seminormal.

Theorem 1.3.
Let X be a p-subgroup of a group G which is seminormal. Then X is subnormal in G.

Examples 4.2 and 4.3 show that if the order of a seminormal subgroup of a group has more than one prime divisor it need not be subnormal.

[1] *Note that the term **seminormal** has several different meanings in the literature.*

Theorem 1.4.
Let G be a solvable group. Then G is an SN-group if and only if it is a PST-group.

Theorem 1.5.
Let G be a solvable group. Then the following are equivalent:

(1) *G is an SN-group;*

(2) *G is an SP-group;*

(3) *G is an SPS-group;*

(4) *G is a PST-group.*

Let p be a prime. As was introduced by Robinson in [13], a group G is called a C_p-group if every subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$. As was introduced by Ballester-Bolinches and Esteban-Romero [5], a group G is called a Y_p-group if whenever K is a p-subgroup of G every subgroup of K is S-permutable in $N_G(K)$.

Theorem 1.6 ([5]).
Let G be a group.

(1) *G is a solvable PST-group if and only if G is a Y_p-group for all primes p.*

(2) *G is Y_p-group if and only if G is either p-nilpotent or G is a C_p-group and has abelian Sylow p-subgroups.*

Definition 1.7.
Let G be a group.

(1) G is a *$\widehat{Y}_p$-group* if for every p-subgroup K of G every subgroup of K is semipermutable in $N_G(K)$.

(2) G is a *$\widetilde{Y}_p$-group* if for every p-subgroup K of G every subgroup of K is S-semipermutable in $N_G(K)$.

(3) G is a $\widetilde{\widetilde{Y}}_p$-group if for every p-subgroup K of G every subgroup of K is seminormal in $N_G(K)$.

It was established in [2, Theorem D] that $Y_p = \widehat{Y}_p = \widetilde{Y}_p$. We arrive at the following result:

Theorem 1.8.
$Y_p = \widehat{Y}_p = \widetilde{Y}_p = \widetilde{\widetilde{Y}}_p$.

Using Theorems 1.6 and 1.8, the following corollary is immediate.

Corollary 1.9.
Let G be a group. The following are equivalent.

(1) *G is a solvable PST-group.*

(2) *G is a Y_p-group for all primes p.*

(3) *G is a $\widehat{Y}_p$-group for all primes p.*

(4) *G is a $\widetilde{Y}_p$-group for all primes p.*

(5) *G is a $\widetilde{\widetilde{Y}}_p$-group for all primes p.*

2. Preliminaries

In this section we present a number of results which are used in proving Theorems 1.1–1.6, 1.8.

Lemma 2.1 ([15]).
Let H be an S-semipermutable subgroup of a group G and let N be a normal p-subgroup of G, p a prime.

(1) *If $H \leq K \leq G$, then H is S-semipermutable in K.*

(2) *HN/N is S-semipermutable in G/N.*

(3) *If all the p'-elements of G act by conjugation as power automorphisms on N, then $G/C_G(N)$ is nilpotent.*

Lemma 2.2.
Let G be a solvable SPS-group.

(1) *If T is a subnormal p-subgroup of G, p a prime, then $O^p(G)$ acts as power automorphisms on T.*

(2) *G is supersolvable.*

Proof. (1) Let x be a nontrivial element of T and let Q be a Sylow q-subgroup of G where $p \neq q$. Then $\langle x\rangle Q = Q\langle x\rangle$ and $\langle x\rangle$ is a subnormal Sylow p-subgroup of $\langle x\rangle Q$. Thus Q normalizes $\langle x\rangle$ and result follows.

(2) Let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group for some prime p. Let P be the Sylow p-subgroup of G and let x be a nontrivial element in $M \cap Z(P)$. By (1), $\langle x\rangle$ is a normal subgroup of G. By (2) of Lemma 2.1, $G/\langle x\rangle$ is an SPS-group and, by induction, it is supersolvable. Hence G is supersolvable. □

Lemma 2.3.
Let G be a solvable PST-group and let M be a subnormal nilpotent subgroup of G. Then M is S-seminormal.

Proof. Assume first that M is a p-group, p a prime. Let L be the nilpotent residual of G. By [1, Theorem 2.3], L is an abelian normal Hall subgroup upon which G acts as power automorphisms.

Assume $M \leq L$. Then M is normal in G and hence S-seminormal. Thus we may assume $M \cap L = 1$. Note $M \leq O_p(G)$ and $O_p(G) \cap L = 1$. Let q be a prime divisor of the order of G and let Q be a Sylow q-subgroup of G, $q \neq p$. Then either $Q \leq L$ or $Q \cap L = 1$. Assume that $Q \leq L$. Then $Q \trianglelefteq G$ and $QM \leq G$. But M is a subnormal Sylow p-subgroup of QM so that $QM = Q{\times}M$. Now assume $Q \cap L = 1$. Then $(QL/L)(ML/L) - MQL/L = QML/L$ since G/L is nilpotent. It follows that $[Q,M] = L$. But $[Q,M] \leq [Q,O_p(G)] \leq O_p(G)$. This means that $[Q,M] \leq L \cap O_p(G) = 1$ and $QM = Q{\times}M$. Hence M is an S-seminormal subgroup of G.

Now assume $|M| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the prime factorization of $|M|$. Let P_i be the Sylow p_i-subgroup of M, $1 \leq i \leq r$. Then P_i is a subnormal S-seminormal subgroup of G.

Let q be a prime divisor of $|G|$ with $q \notin \{p_1, \dots, p_r\}$. Let Q be a Sylow q-subgroup of G. Then Q normalizes P_i and so Q normalizes M. Thus M is an S-seminormal subgroup of G. □

3. Proofs of the main results

Proof of Theorem 1.2. Let H be an S-seminormal subgroup of G and let T be a subgroup of G such that $(|H|,|T|) = 1$. Let $|T| = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the prime factorization of $|T|$ and let R_i be a Sylow p_i-subgroup of T. Further, let P_i be a Sylow p_i-subgroup of G such that $R_i \leq P_i$. Then P_i normalizes H and it follows that T normalizes H. Thus H is seminormal. A seminormal subgroup of G is certainly S-seminormal and so the theorem holds. □

Proof of Theorem 1.3. Let X be a seminormal p-subgroup of a group G, p a prime and let Q be a Sylow q-subgroup of G, $q \neq p$. Then Q normalizes X and so $O^p(G) \leq N_G(X)$. Therefore, $[G{:}N_G(X)]$ is a power of p. By [14, Lemma A], X is S-permutable in G. Hence, by Kegel's result [10], X is subnormal in G. □

Proof of Theorem 1.1. Let G be a solvable SPS-group and let p be the largest prime divisor of the order of G. Let P be a Sylow p-subgroup of G. By (2) of Lemma 2.2, G is supersolvable and so P is a normal subgroup of G.

Let L be the nilpotent residual of G. By [1, Theorem 2.4], it is enough to show L is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms. Note L is nilpotent since G is supersolvable.

By (2) of Lemma 2.1, G/P is an SPS-group and so, by induction, it is a PST-group. By [1, Theorem 2.3], LP/P is an abelian Hall subgroup of G/P. Note that either $G = PC_G(P)$ or $G/C_G(P)$ contains a nontrivial p'-element. Assume first that $G = PC_G(P)$. Then $P \leq Z_*(G)$, where $Z_*(G)$ denotes the hypercenter of G, and $G = P \times Q$ where Q is the Hall p'-subgroup of G. Then $L \leq Q$ and $L \simeq LP/P$ so that L is an abelian Hall subgroup of G. Now assume that $G/C_G(P)$ is not a p-group and let x be a nontrivial p'-element of G. By (1) of Lemma 2.2, x acts as a power automorphism on P so that $P = [P, x]$ and hence $P = [P, G] \leq L$. By (1) of Lemma 2.2 and (3) of Lemma 2.1, $G/C_G(P)$ is nilpotent so that $L \leq C_G(P)$. This means $P \leq Z(L)$. Now L/P is an abelian Hall subgroup of G/P and so L is abelian Hall subgroup of G.

Let y be an element in L of order a power of a prime r. Since $\langle y \rangle$ is subnormal in G and G is an SPS-group, it follows that $O^r(G)$ normalizes $\langle y \rangle$. Thus $\langle y \rangle \trianglelefteq G$ and G acts as power automorphisms on L and G is a PST-group.

Conversely, if G is a solvable PST-group, it is an SPS-group. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.4. Let G be a solvable group. If G is an SN-group, then it is an SPS-group and hence a PST-group by Theorem 1.1.

Assume that G is a PST-group and let X be a subnormal subgroup of G. By Theorem 1.3, it is enough to show X is S-seminormal. Let L be the nilpotent residual of G. By [1, Theorem 2.3], L is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms.

Consider $L \cap X$. Assume first that $L \cap X \neq 1$, set $Y = L \cap X$ and note $Y \trianglelefteq G$. Let q be a prime divisor of $|G|$ such that $(q, |X|) = 1$. Further, let Q be a Sylow q-subgroup of G. Now G/Y is a PST-group, X/Y is a subnormal subgroup of G/Y and $(q, |X/Y|) = 1$. By induction it follows that QY/Y normalizes X/Y in G/Y. Hence Q normalizes X in G. Therefore, we may assume $L \cap X = 1$ and so X is nilpotent. By Lemma 2.3, X is an S-seminormal subgroup of G and G is an SN-group. □

Proof of Theorem 1.5. This follows directly from Theorems 1.1 and 1.4. □

Proof of Theorem 1.8. By [2, Theorem D] it is enough to show $\tilde{\tilde{Y}}_p = Y_p$, p a prime. Let $G \in \tilde{\tilde{Y}}_p$ and let K be a p-subgroup of G. Let H be a subgroup of K and let Q be a Sylow q-subgroup of $N_G(K)$, $q \neq p$. Then $QH = HQ$ since Q normalizes H and thus $G \in Y_p$.

Let $G \in Y_p$. By (2) of Theorem 1.6, G is either p-nilpotent or G is a C_p-group and has abelian Sylow p-subgroups. Let K be a p-subgroup of G.

First assume G is p-nilpotent. Then $N_G(K)$ is p-nilpotent. Let H be a subgroup of K and let Q be a Sylow q-subgroup of $N_G(H)$, $q \neq p$. Since H is a subnormal Sylow p-subgroup of QH, $QH = HXQ$ and hence H is S-seminormal in $N_G(H)$.

Next assume that a Sylow p-subgroup P of G is abelian and every subgroup of P is normal in $N_G(P)$. We may assume $H \leq K \leq P$. Then $H \trianglelefteq N_G(P)$. Let $x \in N_G(K)$ and note that $H \trianglelefteq N_G\left(P^{x^{-1}}\right)$. Thus $H \trianglelefteq \left\langle N_G(P), N_G\left(P^{x^{-1}}\right)\right\rangle$ and since $N_G(P)$ is abnormal in G, $x^{-1} \in \left\langle N_G(P), N_G\left(P^{x^{-1}}\right)\right\rangle$. Thus $H^x = H$ and $H \trianglelefteq N_G(K)$. This means that H is an S-seminormal subgroup of $N_G(K)$ and $\in \tilde{\tilde{Y}}_p$. This completes the proof of Theorem 1.8. □

4. Examples

Example 4.1.

Let S_4, A_4, and K_4 denote, respectively, the symmetric group of order 4, the alternating group of order 4, and the Klein 4-group. Let $G = S_4$ and let $H = \langle(123)\rangle$. Then H is S-semipermutable in G but it is not semipermutable in G since it does not permute with an element of order 2 in K_4, the Sylow 2-subgroup of A_4.

An S-permutable subgroup of a group is subnormal. That this is not the case with S-semipermutable subgroups can be seen in the subgroup H in S_4. Notice that H is not seminormal in S_4.

Example 4.2.

Let $D_{10} = \langle x, y \mid x^5 = y^2 = 1, x^y = x^{-1}\rangle$, the dihedral group of order 10, and $C_{15} = \langle t, s \mid t^5 = s^3 = 1, ts = st\rangle$, the cyclic group of order 15. Let $G = D_{10}\times C_{15}$ and let $K = \langle t\rangle\times\langle y\rangle$. Since $\langle s\rangle$ centralizes K if follows that K is seminormal in G. Note that K is not subnormal in G.

Example 4.3.

Let $H = \langle x\rangle\rtimes\langle y\rangle$ be a semidirect product of a cyclic group, $\langle x\rangle$, of order 11 by a cyclic group, $\langle y\rangle$, of order 5. Let $G = H\times S_4$. Set $K = \langle x\rangle\times S_3$ where S_4 is a copy of the symmetric group on three elements in S_4. Then K is a seminormal subgroup of G which is not subnormal.

5. An application

In [17], Zhang called a group G an S-group if every subgroup of G is either S-semipermutable or abnormal. He established the following characterization of S-groups.

Theorem 5.1 ([17]).

A group G is an S-group if and only if G is precisely one of the following:

(1) *G is a nilpotent group;*

(2) *$G = L\rtimes D$ with D a system normalizer of G and L the nilpotent residual of G, where L is an abelian Hall subgroup of G of odd order satisfying:*

  (2a) *G acts as power automorphisms on L (that is, G is a solvable PST-group);*

  (2b) *if S is a Sylow subgroup of L and y, z are elements of D, where y is a p-element and z is a q-element (p and q distinct primes), then either $S \le C_L(y)$ or $S \le C_L(z)$.*

Moreover, if G has at least one abnormal Sylow subgroup, then D is a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$ and $C_L(P) = 1$.

Note that an S-group is an SPS-group. This follows because any subnormal subgroup of an S-group is not abnormal and hence S-semipermutable. In [17, Lemma 1], Zhang showed that and S-group is solvable. Thus by Theorem 1.1, an S-group is a PST-group. This establishes part (2a) of Theorem 5.1. Now assume that G is an S-group with an abnormal Sylow p-subgroup, say P. By Theorem 5.1, P is a system normalizer of G. Thus semipermutability is a transitive relation in G by [15, Corollary 4].

A group G is called a BT-group if semipermutability is a transitive relation. BT-groups were introduced by Wang, Li, and Wang in [15] where they classified solvable BT-groups. Other characterizations of solvable BT-groups were established in [2].

We now consider a subclass of the class of S-groups. Let us call a group G an $\bar{S}$-group provided that every subgroup of G is either seminormal or abnormal. Since an $\bar{S}$-group is an S-group, it follows from the above discussion that G is a solvable PST-group. We now show that G is in fact a BT-group.

Theorem 5.2.
Let G be an $\bar{S}$-group. Then G is either nilpotent or a system normalizer of G is an abnormal Sylow subgroup of G. In particular, G is a BT-group.

Proof. Let L be the nilpotent residual of G and D a system normalizer of G. As we have already seen G is a solvable PST-group. We note that L is a normal abelian Hall subgroup of G and G is a semidirect product of L and D, $G = L \rtimes D$. Assume that G is not nilpotent so that $L \neq 1$. Let P be a Sylow p-subgroup of D, p a prime. Then P is a Sylow p-subgroup of G. Assume that P is not an abnormal subgroup of G. Since G is an $\bar{S}$-group, P is S-seminormal. Thus $O^p(G)$ normalizes P so that $P \trianglelefteq G$. Hence if every Sylow subgroup of D is not abnormal, then G is nilpotent which is a contradiction. Let us assume P is abnormal. Then $N_G(P) = P$ and $D = P$ since D is nilpotent. By [15, Corollary 3.4], G is a BT-group. □

References

[1] Agrawal R.K., Finite groups whose subnormal subgroups permute with all Sylow subgroups, Proc. Amer. Math. Soc., 1975, 47(1), 77–83

[2] Al-Sharo K.A., Beidleman J.C., Heineken H., Ragland M.F., Some characterizations of finite groups in which semipermutability is a transitive relation, Forum Math., 2010, 22(5), 855–862

[3] Ballester-Bolinches A., Cossey J., Soler-Escrivà X., On a permutability property of subgroups of finite soluble groups, Commun. Contemp. Math., 2010, 12(2), 207–221

[4] Ballester-Bolinches A., Esteban-Romero R., Sylow permutable subnormal subgroups of finite groups II, Bull. Austr. Math. Soc., 2001, 64(3), 479–486

[5] Ballester-Bolinches A., Esteban-Romero R., Sylow permutable subnormal subgroups of finite groups, J. Algebra, 2002, 251(2), 727–738

[6] Beidleman J.C., Heineken H., Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups, J. Group Theory, 2003, 6(2), 139–158

[7] Beidleman J.C., Heineken H., Pronormal and subnormal subgroups and permutability, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 2003, 6(3), 605–615

[8] Beidleman J.C., Heineken H., Ragland M.F., Solvable PST-groups, strong Sylow bases and mutually permutable products, J. Algebra, 2009, 321(7), 2022–2027

[9] Beidleman J.C., Ragland M.F., The intersection map of subgroups and certain classes of finite groups, Ric. Mat., 2007, 56(2), 217–227

[10] Kegel O.H., Sylow-Gruppen und Subnormalteiler endlicher Gruppen, Math. Z., 1962, 78, 205–221

[11] Maier R., Zur Vertauschbarkeit und Subnormalität von Untergruppen, Arch. Math. (Basel), 1989, 53(2), 110–120

[12] Ore O., Contributions to the theory of groups of finite order, Duke Math. J., 1939, 5(2), 431–460

[13] Robinson D.J.S., A note on finite groups in which normality is transitive, Proc. Amer. Math. Soc., 1968, 19(4), 933–937

[14] Schmid P., Subgroups permutable with all Sylow subgroups, J. Algebra, 1998, 207(1), 285–293

[15] Wang L., Li Y., Wang Y., Finite groups in which (S-)semipermutability is a transitive relation, Int. J. Algebra, 2008, 2(3) 143–152

[16] Zacher G., I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 1964, 37, 150–154

[17] Zhang Q., s-semipermutability and abnormality in finite groups, Comm. Algebra, 1999, 27(9), 4515–4524

[18] Zhang Q.H., Wang L.F., The influence of s-semipermutable subgroups on finite groups, Acta Math. Sinica (Chin. Ser.), 2005, 48(1), 81–88 (in Chinese)