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Results on the deficiencies of some differential-difference polynomials of meromorphic functions

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Abstract: In this paper, we study the relation between the deficiencies concerning a meromorphic function $f(z)$, its derivative $f'(z)$ and differential-difference monomials $f(z)^m f(z+c)f'(z)$, $f(z+c)^n f'(z)$, $f(z)^m f(z+c)$. The main results of this paper are listed as follows: Let $f(z)$ be a meromorphic function of finite order satisfying

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f)}{T(r, f')} < +\infty,$$

and c be a non-zero complex constant, then $\delta(\infty, f(z)^m f(z+c)f'(z)) \geq \delta(\infty, f')$ and $\delta(\infty, f(z+c)^n f'(z)) \geq \delta(\infty, f')$. We also investigate the value distribution of some differential-difference polynomials taking small function $a(z)$ with respect to $f(z)$.

Keywords: Meromorphic function, Differential-difference polynomial, Deficiency

MSC: 30D35, 39A10

1 Introduction and main results

The fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used (see Hayman [1], Yang [2] and Yi-Yang [3]). In addition, for a meromorphic function $f(z)$, we use $\delta(a, f)$ to denote the Nevanlinna deficiency of $a \in \widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where

$$\delta(a, f) = \liminf_{r \rightarrow +\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r \rightarrow +\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

We also use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r \rightarrow \infty} \int_{[1, r) \cap E} \frac{dt}{t} < \infty$.

Throughout this paper, we assume m, n, k, t are positive integers.

Many people were interested in the value distribution of different expressions of meromorphic functions and obtained lots of important theorems (see [1, 4–6]).

Recently, the topic of complex differences has attracted the interest of many mathematicians, and a number of papers have focused on the value distribution of complex differences and difference analogues of Nevanlinna theory (including [7–11]). By combining complex differentiates and complex differences, we proceed in this way in this paper.

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Firstly, we study the Nevanlinna deficiencies related to a meromorphic function $f(z)$, its derivative $f'(z)$ and its differential-difference monomials

$$F_1(z) = f(z)^m f(z+c)f'(z) \quad \text{and} \quad F_2(z) = f(z+c)^n f'(z),$$

and obtain the following theorems.

Theorem 1.1. *Let $f(z)$ be a meromorphic function of finite order satisfying*

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f)}{T(r, f')} < +\infty, \tag{1}$$

and c be a non-zero complex constant. Then

$$\delta(\infty, F_1) \geq \delta(\infty, f').$$

Theorem 1.2. *Let $f(z)$ be a meromorphic function of finite order satisfying (1), and c be a non-zero complex constant. Then*

$$\delta(\infty, F_2) \geq \delta(\infty, f').$$

Example 1.3. *It is easy to find meromorphic functions to make the inequalities in Theorems 1.1 and 1.2 hold. For example, let $f_1(z) = e^z$, then*

$$\delta(\infty, f_1') = \delta(\infty, F_1) = \delta(\infty, F_2) = \delta(\infty, f_1) = 1,$$

showing the equalities in Theorems 1.1 and 1.2 may hold.

Let $f_2(z) = \frac{z^3+2}{z}$ and $c = 1 = m = n$, then

$$f_2'(z) = \frac{2z^3 - 2}{z^2} \quad \text{and} \quad f_2(z+1) = \frac{z^3 + 3z^2 + 3z + 3}{z+1}.$$

It follows that

$$F_1(z) = f_2(z)f_2(z+1)f_2'(z) = \frac{2z^9 + P_1(z)}{z^4 + z^3}, \quad F_2(z) = f_2(z+1)f_2'(z) = \frac{2z^6 + P_2(z)}{z^3 + z^2},$$

where $P_1(z), P_2(z)$ are polynomials in z with $\deg P_1(z) \leq 8$ and $\deg P_2(z) \leq 5$. Clearly, $f_2(z)$ satisfies

$$\limsup_{r \rightarrow +\infty} \frac{T(r, f_2)}{T(r, f_2')} = \limsup_{r \rightarrow +\infty} \frac{3 \log r}{3 \log r} = 1 < +\infty.$$

Thus, we have

$$\delta(\infty, F_1) = \frac{5}{9} > \delta(\infty, f_2') = \frac{1}{3},$$

and

$$\delta(\infty, F_2) = \frac{1}{2} > \delta(\infty, f_2') = \frac{1}{3},$$

showing the inequalities in Theorems 1.1 and 1.2 may hold.

Thus, Theorems 1.1 and 1.2 are sharp.

In addition, from the above examples, we can find that $\delta(\infty, F_1) = \frac{5}{9} < \delta(\infty, f_2) = \frac{2}{3}$ and $\delta(\infty, F_2) = \frac{1}{2} < \delta(\infty, f_2) = \frac{2}{3}$. So, we give the following question: Under the conditions of Theorems 1.1 and 1.2, do F_1, F_2 satisfy

$$\delta(\infty, f') \leq \delta(\infty, F_1) \leq \delta(\infty, f),$$

and

$$\delta(\infty, f') \leq \delta(\infty, F_2) \leq \delta(\infty, f)?$$

We also get the following relations between $\delta(\infty, f)$ and $\delta(\infty, F_i)$, $i = 1, 2$.

Theorem 1.4. *Let $f(z)$ be a meromorphic function of finite order, and c be a non-zero complex constant. If $\delta = \delta(\infty, f) > \frac{8}{m+6}$, then $\delta(\infty, F_1) > 0$.*

Theorem 1.5. *Let $f(z)$ be a meromorphic function of finite order, and c be a non-zero complex constant. If $\delta = \delta(\infty, f) > \frac{8}{m+5}$, then $\delta(\infty, F_2) > 0$.*

Theorem 1.6. *Let $f(z)$ be a meromorphic function of finite order, and c be a non-zero complex constant. Set*

$$F_3(z) = f(z)^m f(z+c).$$

If $\delta = \delta(\infty, f) > \frac{4}{m+3}$, then $\delta(\infty, F_3) > 0$.

Remark 1.7. *From the conclusions of Theorems 1.1, 1.4, 1.5 and 1.6, we see that there may exist some meromorphic function $f(z)$ satisfying $\delta(\infty, F_i) = 0$, $i = 1, 2, 3$ as $\delta(\infty, f) > 0$. An interesting problem arises naturally: How can we find some meromorphic function $f(z)$ to satisfy $\delta(\infty, F_i) = 0$, $i = 1, 2, 3$ as $\delta(\infty, f) > 0$?*

The following ideas derive from Hayman [5], Laine-Yang [12], Zheng-Chen [13]. In 1959, Hayman [5] studied the value distribution of meromorphic functions and their derivatives, and obtained the following famous theorems.

Theorem 1.8 ([5]). *Let $f(z)$ be a transcendental entire function. Then*

- (i) *for $n \geq 3$ and $a \neq 0$, $\Psi(z) = f'(z) - af(z)^n$ assumes all finite values infinitely often.*
- (ii) *for $n \geq 2$, $\Phi(z) = f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.*

Recently, some authors studied the zeros of $f(z+c)f(z)^n - a$ and $f(z+c) - af(z)^n - b$, where $a (\neq 0)$, b are complex constants or small functions. Some related results can be found in [12–17]. Especially, Laine-Yang [12] and Zheng-Chen [13] proved the following result, which is regarded as a difference counterpart of Theorem 1.8.

Theorem 1.9 ([12, 13]). *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then*

- (i) *for $n \geq 2$, $\Phi_1(z) = f(z+c)f(z)^n$ assumes every $a \in \mathbb{C} \setminus \{0\}$ infinitely often.*
- (ii) *for $n \geq 3$ and $a \neq 0$, $\Psi_1(z) = f(z+c) - af(z)^n$ assumes every $b \in \mathbb{C}$ infinitely often.*

In the following, we investigate the zeros of some differential-difference polynomials of a meromorphic function $f(z)$ taking small function $a(z)$ with respect to $f(z)$, where and in the following $a(z)$ is a non-zero small function of growth $S(r, f)$, and obtain some theorems as follows.

Theorem 1.10. *Let $f(z)$ be a transcendental meromorphic function of finite order, and c be a non-zero complex constant. Set*

$$G_1(z) = f(z)^m f(z+c)^n f'(z).$$

If $m \geq n + 8$ or $n \geq m + 8$, then $G_1(z) - a(z)$ has infinitely many zeros.

Example 1.11. *An example shows that the conclusion can not hold if $f(z)$ is of infinite order. Let $f(z) = 2e^{e^z}$, $a(z) = e^z$, $m = 9$, $n = 1$ and $e^c = -10$, then*

$$G_1(z) - a(z) = (2^{m+n+1} e^{[(m+1)+ne^c]e^z} - 1)e^z = (2^{11} - 1)e^z,$$

then $G_1(z) - a(z)$ has finite many zeros.

Theorem 1.12. *Let $f(z)$ be a transcendental meromorphic function of finite order, and c_1, c_2, \dots, c_n be non-zero complex constants. Set*

$$G_2(z) = f(z)^m \prod_{j=1}^n f(z+c_j)^{s_j} \prod_{i=1}^k f^{(i)}(z).$$

If $m \geq \sigma_1 + 2n + k(k+3) + 4$, where $\sigma_1 = s_1 + s_2 + \dots + s_n$, then $G_2(z) - a(z)$ has infinitely many zeros.

Let $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a non-zero polynomial, where a_0, a_1, \dots, a_n are complex constants and t is the number of the distinct zeros of $P_n(z)$. Then we further obtain the following results.

Theorem 1.13. Let $f(z)$ be a transcendental meromorphic function of finite order, and c be a non-zero complex constant. Set

$$G_3(z) = f(z)^m P_n(f(z+c)) \prod_{i=1}^k f^{(i)}(z).$$

If $m \geq n + t + k(k+3) + 4$, then $G_3(z) - a(z)$ has infinitely many zeros.

Theorem 1.14. Let $f(z)$ be a transcendental meromorphic function of finite order, and c be a non-zero complex constant. Set

$$G_4(z) = P_m(f(z)) f(z+c)^n \prod_{i=1}^k f^{(i)}(z).$$

If $m \geq n + t + k(k+3) + 4$, then $G_4(z) - a(z)$ has infinitely many zeros.

2 Some lemmas

To prove the above theorems, we will require some lemmas as follows.

Lemma 2.1 ([7, 10]). Let $f(z)$ be a meromorphic function of finite order ρ and c be a fixed non-zero complex number, then we have

$$m \left(r, \frac{f(z+c)}{f(z)} \right) + m \left(r, \frac{f(z)}{f(z+c)} \right) = S(r, f).$$

By [18], [19, p.66] and [20], we immediately deduce the following lemma.

Lemma 2.2. Let $f(z)$ be a meromorphic function of finite order, and c be a non-zero complex constant. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

$$N(r, f(z+c)) = N(r, f) + S(r, f), \quad N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.3 ([3, p.37]). Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then

$$N(r, f^{(l)}) = N(r, f) + l\bar{N}(r, f), \quad T(r, f^{(l)}) \leq T(r, f) + l\bar{N}(r, f) + S(r, f).$$

Lemma 2.4. Let $f(z)$ be a transcendental meromorphic function of finite order, and $G_1(z) = f(z)^m f(z+c)^n \cdot f'(z)$. Then we have

$$(|m-n|-1)T(r, f) + S(r, f) \leq T(r, G_1) \leq (n+m+2)T(r, f) + S(r, f). \quad (2)$$

Proof. From Lemmas 2.2 and 2.3, we have

$$T(r, G_1) \leq T(r, f^m) + T(r, f(z+c)^n) + T(r, f') \leq (n+m+2)T(r, f) + S(r, f).$$

On the other hand, from Lemma 2.2 again, we have

$$(n+m+1)T(r, f) = T(r, f^{n+m+1}) = T \left(r, \frac{f(z)^{n+1} G_1(z)}{f(z+c)^n f'(z)} \right)$$

$$\begin{aligned} &\leq T(r, G_1) + T\left(r, \frac{f}{f'}\right) + T\left(r, \frac{f(z)^n}{f(z+c)^n}\right) \\ &\leq T(r, G_1) + (2n + 2)T(r, f) + S(r, f), \end{aligned}$$

where we assume $m \geq n$ without loss of generality. Thus, (2) is proved. □

Using the similar method as in Lemma 2.4, we get the following lemmas.

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function of finite order, and $G_2(z) = f(z)^m \prod_{j=1}^n f(z + c_j)^{s_j} \prod_{i=1}^k f^{(i)}(z)$, then we have*

$$\left(m - \sigma_1 - \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f) \leq T(r, G_2) \leq \left(m + \sigma_1 + \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f).$$

Lemma 2.6. *Let $f(z)$ be a transcendental meromorphic function of finite order, and $G_3(z) = f(z)^m P_n(f(z + c)) \prod_{i=1}^k f^{(i)}(z)$, then we have*

$$\left(m - n - \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f) \leq T(r, G_3) \leq \left(m + n + \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f).$$

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of finite order, and $G_4(z) = P_m(f(z))f(z + c)^n \prod_{i=1}^k f^{(i)}(z)$, then we have*

$$\left(m - n - \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f) \leq T(r, G_4) \leq \left(m + n + \frac{k(k + 3)}{2}\right) T(r, f) + S(r, f).$$

3 Proofs of Theorems 1.1 and 1.2

3.1 Proof of Theorem 1.1

We firstly give the following elementary inequalities

$$\frac{\alpha}{\alpha + \beta} \leq \frac{\alpha_1}{\alpha_1 + \beta}, \quad \frac{\alpha}{\alpha + \beta} \leq \frac{\alpha + \gamma}{\alpha + \beta + \gamma}, \tag{3}$$

for $\alpha, \beta, \gamma \geq 0$ and $\alpha \leq \alpha_1$.

Since

$$f'(z)^{m+2} = F_1(z) \frac{f'(z)^{m+1}}{f(z)^{m+1}} \frac{f(z)}{f(z+c)},$$

it follows that

$$(m + 2)m(r, f') \leq m(r, F_1) + (m + 1)m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right).$$

Then by Lemma 2.1, we have

$$m(r, F_1) \geq (m + 2)m(r, f') + S(r, f). \tag{4}$$

Since $N(r, f') = N(r, f) + \overline{N}(r, f)$, it follows by Lemma 2.2 that

$$N(r, F_1) \leq (m + 2)N(r, f) + \overline{N}(r, f) \leq (m + 2)N(r, f'). \tag{5}$$

From (1), we have

$$\limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f')} = \limsup_{r \rightarrow +\infty} \frac{S(r, f)}{T(r, f)} \frac{T(r, f)}{T(r, f')} = 0. \tag{6}$$

Then, from (3)-(6), we have

$$\frac{N(r, F_1)}{T(r, F_1)} \leq \frac{(m + 2)N(r, f')}{(m + 2)N(r, f') + (m + 2)m(r, f') + S(r, f)}$$

$$\leq \frac{N(r, f')}{T(r, f') + S(r, f)} = \frac{N(r, f')}{(1 + o(1))T(r, f')}.$$

It follows that $\delta(\infty, f') \leq \delta(\infty, F_1)$.

Thus, we complete the proof of Theorem 1.1.

3.2 Proof of Theorem 1.2

Since

$$f'(z)^{n+1} = F_2 \frac{f'(z)^n}{f(z)^n} \frac{f(z)^n}{f(z+c)^n},$$

then by using the similar method as in the proof of Theorem 1.1, we can prove Theorem 1.2 easily.

4 Proofs of Theorems 1.4, 1.5 and 1.6

4.1 Proof of Theorem 1.4

Let $F_4(z) = f(z)^{m+2}$, then we have

$$N(r, F_4) = (m + 2)N(r, f) \quad \text{and} \quad T(r, F_4) = (m + 2)T(r, f).$$

It follows that $\delta(\infty, F_4) = \delta(\infty, f) = \delta$. Since $F_4(z) = f(z)^{m+2}$, we have

$$\bar{N}(r, f) \leq N(r, f) \leq \frac{1}{m + 2}N(r, F_4) \leq \frac{1 - \delta}{m + 2}T(r, F_4) + S(r, F_4), \tag{7}$$

$$\bar{N}(r, \frac{1}{f}) \leq N(r, \frac{1}{f}) \leq \frac{1}{m + 2}N(r, \frac{1}{F_4}) \leq \frac{1}{m + 2}T(r, F_4) + O(1). \tag{8}$$

From (7), (8) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned} T(r, F_4) &= T\left(r, F_1(z) \frac{f(z)}{f(z+c)} \frac{f(z)}{f'(z)}\right) \\ &\leq T(r, F_1) + N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f}) + N(r, f) + \bar{N}(r, f) + S(r, f) \\ &\leq T(r, F_1) + \frac{4 - 2\delta}{m + 2}T(r, F_4) + S(r, f), \end{aligned}$$

that is

$$T(r, F_1) \geq \left(\frac{m - 2(1 - \delta)}{m + 2} + o(1)\right)T(r, F_4). \tag{9}$$

From (7) and (8) again, we have

$$\begin{aligned} N(r, F_1) &\leq N(r, F_4) + N(r, \frac{f(z+c)}{f(z)}) + N(r, \frac{f'}{f}) \\ &\leq N(r, F_4) + N(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f}) + N(r, f) + \bar{N}(r, f) + S(r, f) \\ &\leq \frac{(m + 4)(1 - \delta) + 2}{m + 2}T(r, F_4) + S(r, f), \end{aligned}$$

that is,

$$N(r, F_1) \leq \left(\frac{(m + 4)(1 - \delta) + 2}{m + 2} + o(1)\right)T(r, F_4). \tag{10}$$

Thus, from (9), (10) and $\delta = \delta(\infty, f) = \delta(\infty, F_4) > \frac{8}{m+6}$, it follows that

$$\limsup_{r \rightarrow +\infty} \frac{N(r, F_1)}{T(r, F_1)} \leq \frac{(m + 4)(1 - \delta) + 2}{m - 2(1 - \delta)} < 1,$$

that is,

$$\delta(\infty, F_1) = 1 - \limsup_{r \rightarrow +\infty} \frac{N(r, F_1)}{T(r, F_1)} > 0.$$

This completes the proof of Theorem 1.4.

4.2 Proofs of Theorems 1.5 and 1.6

Using the similar method as in the proof of Theorem 1.4, we can prove Theorems 1.5 and 1.6 easily.

5 Proofs of Theorems 1.10, 1.12, 1.13 and 1.14

5.1 Proof of Theorem 1.10

Suppose that $f(z)$ is a transcendental meromorphic function of finite order. Since $m \geq n + 8$ or $n \geq m + 8$, then by Lemma 2.4, we have $S(r, f) = S(r, G_1)$. Thus, by using the second fundamental theorem and Lemmas 2.2 and 2.4, we have

$$\begin{aligned} & (|m - n| - 1)T(r, f) \leq T(r, G_1) + S(r, f) \\ & \leq \bar{N}(r, G_1) + \bar{N}\left(r, \frac{1}{G_1}\right) + \bar{N}\left(r, \frac{1}{G_1(z) - a(z)}\right) + S(r, G_1) \\ & \leq \bar{N}(r, f(z)) + \bar{N}(r, f(z + c)) + \bar{N}\left(r, \frac{1}{f(z)}\right) \\ & \quad + \bar{N}\left(r, \frac{1}{f(z + c)}\right) + \bar{N}\left(r, \frac{1}{f'}\right) + \bar{N}\left(r, \frac{1}{G_1(z) - a(z)}\right) + S(r, G_1) \\ & \leq 6T(r, f) + \bar{N}\left(r, \frac{1}{G_1(z) - a(z)}\right) + S(r, G_1), \end{aligned}$$

that is

$$\frac{|m - n| - 7}{n + m + 2} T(r, G_1) + S(r, G_1) \leq (|m - n| - 7)T(r, f) \leq \bar{N}\left(r, \frac{1}{G_1(z) - a(z)}\right) + S(r, G_1).$$

Thus, we have from $m \geq n + 8$ or $n \geq m + 8$ that

$$\delta(a, G_1) \leq 1 - \frac{|m - n| - 7}{n + m + 2} < 1.$$

Consequently, $G_1(z) - a(z)$ has infinitely many zeros.

This completes the proof of Theorem 1.10.

5.2 Proof of Theorem 1.14

If $f(z)$ is a transcendental meromorphic function of finite order, then by Lemma 2.7, we have $S(r, f) = S(r, G_4)$. Thus, by using the second fundamental theorem and Lemmas 2.2 and 2.7 again,

$$\begin{aligned} & \left(m - n - \frac{k(k + 3)}{2}\right)T(r, f) \leq T(r, G_4) + S(r, f) \\ & \leq \bar{N}(r, G_4) + \bar{N}\left(r, \frac{1}{G_4}\right) + \bar{N}\left(r, \frac{1}{G_4(z) - a(z)}\right) + S(r, G_4) \\ & \leq \bar{N}(r, f) + \bar{N}(r, f(z + c)) + \sum_{i=1}^t \bar{N}\left(r, \frac{1}{f - \gamma_i}\right) + \bar{N}\left(r, \frac{1}{f(z + c)}\right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^k \overline{N}\left(r, \frac{1}{f^{(j)}}\right) + \overline{N}\left(r, \frac{1}{G_4(z) - a(z)}\right) + S(r, G_4) \\
& \leq \left(3 + t + \frac{k(k+1)}{2} + k\right) T(r, f) + \overline{N}\left(r, \frac{1}{G_4(z) - a(z)}\right) + S(r, G_4),
\end{aligned}$$

that is,

$$\begin{aligned}
& \frac{m - n - t - k(k+3) - 3}{m + n + \frac{k(k+3)}{2}} T(r, G_4) + S(r, G_4) \\
& \leq (m - n - t - k(k+3) - 3) T(r, f) \leq \overline{N}\left(r, \frac{1}{G_4(z) - a(z)}\right) + S(r, G_4),
\end{aligned}$$

where $\gamma_1, \gamma_2, \dots, \gamma_t$ are distinct zeros of $P_m(z)$. Since $m \geq n + t + k(k+3) + 4$, we have

$$\delta(a, G_4) \leq 1 - \frac{m - n - t - k(k+3) - 3}{m + n + \frac{k(k+3)}{2}} < 1.$$

Consequently, $G_4(z) - a(z)$ has infinitely many zeros.

This completes the proof of Theorem 1.14.

5.3 Proofs of Theorems 1.12 and 1.13

Using the similar method as in the proofs of Theorems 1.10 and 1.14 and combining Lemmas 2.5 and 2.6, we can prove Theorems 1.12 and 1.13 easily.

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