

Arbitrary l -state approximate solutions of the Hulthén potential through the exact quantization rule

Research Article

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Abstract:

In this paper, using the Exact Quantization Rule, we present approximate analytical solutions of the radial Schrödinger equation with non-zero l values for the Hulthén potential in the frame of an approximation to the centrifugal potential for any l states. The energy levels of all bound states can be easily calculated from the Exact Quantization Rule. Specifically, the normalized analytical wave functions are also obtained. Some energy eigenvalues are numerically calculated and compared with those obtained by other methods such as asymptotic iteration, supersymmetry, numerical integration methods, and the *schroedinger* Mathematica package.

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1. Introduction

Since quantum mechanics was first established, the finding of exact solutions has been an important research subject which has attracted much attention in this field, because the exact wave function contains all the necessary information about the quantum system under consideration. Over the last few decades a variety of analytical methods such as the supersymmetry method (SUSY) [1], Nikiforov-Uvarov (NU) method [2] and asymptotic iteration method (AIM) [3, 4] have been developed to solve

the wave equations exactly for physically interesting potentials. Recently, an alternative method for solving the one-dimensional Schrödinger equation, and the three-dimensional radial Schrödinger equation was developed by Ma and Xu [5, 6].

Ma and Xu [5] have used this new method, which is called exact quantization rule (EQR), to find the exact energy eigenvalues for familiar one-dimensional potentials, as well as the three-dimensional Coulomb potential and the harmonic oscillator potential. By employing this approach, Qiang and Dong [7] have also found arbitrary l -state solutions of the rotating Morse potential with the Pekeris approximation. Moreover, using this method, Qiang *et al.* have also obtained the relativistic solution for the rotational Morse potential with pseudospin symmetry

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for any κ state [8]. These results show that the Exact Quantization Rule can be efficiently used to obtain the exact bound-state solutions for most solvable potentials. On the other hand, it is well known that the exact solutions of the Schrödinger equation can be obtained for only a few particular forms of potentials. Some physically interesting potentials can not be solved exactly, while the centrifugal term in the three-dimensional Schrödinger equation prevents exact solutions to other potentials (even though they can be solved exactly in the case of the one-dimensional Schrödinger equation). In these cases, approximate methods have been developed to find the bound-state energy eigenvalues numerically or quasi-analytically. For example, Bayrak *et al.* have recently obtained l -state solutions for the Morse and Hulthén potentials using both approximations and the AIM method [9–12]. Using the same technique, Bayrak and Boztosun have also found the relativistic solution of the rotational Morse potential with pseudospin symmetry for any κ -state [13]. The Hulthén potential [14] is one of the most important short-range potentials in physics. It is used extensively to describe the bound and continuum states of the interaction systems, and has been applied to a number of areas such as nuclear and particle physics, atomic physics, and condensed matter and chemical physics [15]. Due to the importance of this potential and the novel power of EQR, in this work we attempt to extend EQR to obtain solutions for any l -state of the Hulthén potential, with some approximation.

This paper is organized as follows: In Section 2 we review the EQR method. In Section 3 this method is applied to obtain the exact eigenvalues of the Hulthén potential. The corresponding normalized wave functions are calculated in Section 4. Some numerical results of energy eigenvalues are presented in Section 5. Finally, some concluding remarks are given in Section 6.

2. Overview of the exact quantization rules

What follows is a brief review of the Exact Quantization Rule.

Ma and Xu have proved that the one-dimensional Schrödinger equation [5, 6]

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2M}{\hbar^2}[E - V(x)]\psi(x) \quad (1)$$

can be written as

$$\frac{d}{dx}\phi(x) = -\frac{2M}{\hbar^2}[E - V(x)] - \phi(x)^2, \quad (2)$$

where $\phi(x) = \psi(x)^{-1}d\psi(x)/dx$ is the logarithmic derivative of the wave function $\psi(x)$, M represents the mass of the particle, and the potential $V(x)$ is a piecewise continuous real function of x . Furthermore, it is known from the Sturm-Liouville theorem that $\phi(x)$ decreases monotonically with respect to x between two turning points, where $E \geq V(x)$. Specifically, x increases across a node of the wave function $\psi(x)$, where $E \geq V(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. By carefully studying the one-dimensional Schrödinger equation, Ma and Xu proposed an Exact Quantization Rule

$$\int_{x_A}^{x_B} k(x)dx = N\pi + \int_{x_A}^{x_B} \phi(x) \left[\frac{dk(x)}{dx} \right] \left[\frac{d\phi(x)}{dx} \right]^{-1} dx, \quad (3)$$

where $k(x) = \sqrt{2M[E - V(x)]}/\hbar$, x_A and x_B are two turning points determined by $E = V(x)$ and $x_A < x_B$, N is the number of nodes of $\phi(x)$ in the region $E \geq V(x)$ and it is larger by one than the number of nodes of the wave function $\psi(x)$. The first term, $N\pi$, relates to the contribution from the nodes of the wave function, and the second term is called the quantum correction. Ma and Xu [5] have found that the quantum correction is independent of the number of nodes for the exactly solvable systems, so it can be replaced in (3) by

$$\int_{x_{0A}}^{x_{0B}} \phi_0(x) \left[\frac{dk_0(x)}{dx} \right] \left[\frac{d\phi_0(x)}{dx} \right]^{-1} dx, \quad (4)$$

where the subscription "0" denotes the ground state. Finally, the Exact Quantization Rule takes the form

$$\int_{x_A}^{x_B} k(x)dx = N\pi + \int_{x_{0A}}^{x_{0B}} \phi_0(x) \left[\frac{dk_0(x)}{dx} \right] \left[\frac{d\phi_0(x)}{dx} \right]^{-1} dx. \quad (5)$$

The quantization rule expressed by (5) can be easily generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential. After separation of the angular part of the wave function, $\psi(r) = r^{-1}R(r)Y_m^l(\theta, \phi)$, the radial part of the Schrödinger equation is

$$\begin{aligned} \frac{d^2}{dr^2}R(r) &= -\frac{2M}{\hbar^2}[E - U(r)]R(r), \\ U(r) &= \frac{l(l+1)\hbar^2}{2Mr^2} + V(r). \end{aligned} \quad (6)$$

Since (6) is similar to (1), the quantization rule (5) is naturally generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential as follows,

$$\int_{r_A}^{r_B} k(r)dr = N\pi + \int_{r_{0A}}^{r_{0B}} \phi_0(r) \left[\frac{dk_0(r)}{dr} \right] \left[\frac{d\phi_0(r)}{dr} \right]^{-1} dr. \quad (7)$$

3. Calculation of the energy eigenvalues by the exact quantization rules

The Hulthén potential we examine in this paper is defined as

$$V_H(r) = -Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \quad (8)$$

where Z and δ are, respectively, the atomic number and the screening parameter, determining the range for the Hulthén potential [14]. The Hulthén potential behaves like the Coulomb potential near the origin ($r \rightarrow 0$), but in the asymptotic region ($r \gg 1$) it decreases exponentially, so its capacity for bound states is smaller than that of the Coulomb potential. However, for small values of the screening parameter δ , the Hulthén potential becomes the Coulomb potential given by $V_c = -\frac{Ze^2}{r}$. The radial part of the Schrödinger equation for Hulthén potential can be expressed as (10)

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R(r) = -\frac{2M}{\hbar^2} [E - V_{\text{eff}}(r)] R(r), \quad (9)$$

where $V_{\text{eff}}(r)$ is the effective potential,

$$V_{\text{eff}}(r) = V_l(r) + V_H(r) = \frac{l(l+1)\hbar^2}{2Mr^2} - Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}}. \quad (10)$$

Letting $R(r) = u(r)/r$ and inserting it into (9) it follows that

$$\frac{d^2 u(r)}{dr^2} = -\frac{2M}{\hbar^2} [E - V_{\text{eff}}(r)] u(r). \quad (11)$$

Generally speaking, there is no analytical solution to (11) for $l \neq 0$ (with an appropriate effective potential given by (10)), therefore an approximation must be made.

According to [10], $\frac{1}{r^2} = \delta^2 \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2}$ can be used for the centrifugal term and the effective potential is finally written as follows,

$$\tilde{V}_{\text{eff}}(r) = \frac{l(l+1)\hbar^2\delta^2}{2M} \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} - Ze^2\delta \frac{e^{-\delta r}}{1 - e^{-\delta r}}. \quad (12)$$

Instead of using the effective potential $V_{\text{eff}}(r)$ given by (10), we use its approximate partner $\tilde{V}_{\text{eff}}(r)$ given by (12) to solve the radial equation for the Hulthén potential. Inserting this new effective potential into equation (11) and using the following notations to make the differential equation more compact,

$$\epsilon^2 = -\frac{2EM}{\hbar^2\delta^2}, \quad \beta^2 = \frac{2MZe^2}{\hbar^2\delta}, \quad \delta r = x, \quad (13)$$

the radial Schrödinger equation takes the following form

$$\frac{d^2 u(x)}{dx^2} + \left[-\epsilon^2 + \beta^2 \frac{e^{-x}}{1 - e^{-x}} - l(1+l) \frac{e^{-x}}{(1 - e^{-x})^2} \right] u(x) = 0, \quad (14)$$

which can further be written as

$$\frac{d^2 u(x)}{dx^2} + k(x)^2 u(x) = 0, \quad (15)$$

where

$$k(x) = \sqrt{-\epsilon^2 - \tilde{V}_{\text{eff}}(x)}, \quad (16)$$

$$\tilde{V}_{\text{eff}}(x) = l(1+l) \frac{e^{-x}}{(1 - e^{-x})^2} - \beta^2 \frac{e^{-x}}{1 - e^{-x}}.$$

Now, we can apply EQR in (7) to the above equation. For this purpose, we first solve the Riccati equation,

$$\frac{d}{dx} \phi_0(x) = -k_0(x)^2 - \phi_0(x)^2. \quad (17)$$

Introducing a new variable of the form $y = \frac{e^{-x}}{1 - e^{-x}}$ and using (13), we obtain

$$y(1+y)\phi_0'(y) + l(1+l)y(1+y) - \beta^2 y + \epsilon_0^2 - \phi_0(y)^2 = 0. \quad (18)$$

We noticed $r \in (0, \infty)$, $x \in (0, \infty)$ and $y \in (\infty, 0)$. Therefore, while $\phi_0(x)$ decreases as x increases, $\phi_0(y)$ increases as y increases. The solution with one node and no pole only has the form $\phi_0(y) = ay + b$, where $a > 0$ due to the monotonic property. Substituting this trial solution into the Riccati equation (17), we get

$$a = 1 + l, \quad b = \frac{(1+l)^2 - \beta^2}{2(1+l)}, \quad \epsilon_0 = \pm \frac{\beta^2 - (1+l)^2}{2(1+l)}. \quad (19)$$

Hence

$$\phi_0(y) = (1+l)y + \frac{(1+l)^2 - \beta^2}{2(1+l)}. \quad (20)$$

In order to calculate the quantum correction, we write $k_0(y)$ as follows for simplicity,

$$k_0(y) = \sqrt{l(1+l)} \sqrt{(y_{0a} - y)(y - y_{0b})}, \quad (21)$$

where $y_{0a} = \frac{e^{-x_{0a}}}{1 - e^{-x_{0a}}}$, $y_{0b} = \frac{e^{-x_{0b}}}{1 - e^{-x_{0b}}}$, x_{0a} and x_{0b} are two turning points and $x_{0a} < x_{0b}$,

$$y_{0a} = \frac{1}{2l(1+l)^2} \left\{ (1+l)[\beta^2 - l(1+l)] + \sqrt{(1+l)[\beta^4 - l(1+l)^3]} \right\},$$

$$y_{0b} = \frac{1}{2l(1+l)^2} \left\{ (1+l)[\beta^2 - l(1+l)] - \sqrt{(1+l)[\beta^4 - l(1+l)^3]} \right\}. \quad (22)$$

We can now calculate the quantum correction,

Table 1. Energy eigenvalues of the Hulthén potential as a function of the screening parameter for 2p, 3p and 3d states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$.

State	δ	EQR [5, 6]	AIM [10]	SUSY [16]	Numerical [15]	schroedinger [17]
2p	0.025	0.1128125	0.1128125	0.1127605	0.1127605	0.1127604
	0.050	0.1012500	0.1012500	0.1010425	0.1010425	0.1010424
	0.075	0.0903125	0.0903125	0.0898478	0.0898478	0.0898477
	0.100	0.0800000	0.0800000	0.0791794	0.0791794	0.0791794
	0.150	0.0612500	0.0612500	0.0594415	0.0594415	0.0594415
	0.200	0.0450000	0.0450000	0.0418854	0.0418860	0.0418860
	0.250	0.0312500	0.0312500	0.0266060	0.0266111	0.0266110
	0.300	0.0200000	0.0200000	0.0137596	0.0137900	0.0137900
	0.350	0.0112500	0.0112500	0.0036146	0.0037931	0.0037931
3p	0.025	0.0437590	0.0437590	0.0437068	0.0437069	0.0437069
	0.050	0.0333681	0.0333681	0.0331632	0.0331645	0.0331645
	0.075	0.0243837	0.0243837	0.0239331	0.0239397	0.0239397
	0.100	0.0168056	0.0168056	0.0160326	0.0160537	0.0160537
	0.150	0.0058681	0.0058681	0.0043599	0.0044663	0.0044663
3d	0.025	0.0437587	0.0437587	0.0436030	0.0436030	0.0436030
	0.050	0.0333681	0.0333681	0.0327532	0.0327532	0.0327532
	0.075	0.0243837	0.0243837	0.0230306	0.0230307	0.0230307
	0.100	0.0168055	0.0168055	0.0144832	0.0144842	0.0144842
	0.150	0.0058681	0.0058681	0.0132820	0.0139660	0.0139658

$$\begin{aligned}
 \int_{x_{0a}}^{x_{0b}} \phi_0(x) \frac{dk_0(x)}{dx} \left(\frac{d\phi_0(x)}{dx} \right)^{-1} dx &= \int_{y_{0a}}^{y_{0b}} \phi_0(y) \frac{dk_0(y)}{dy} \left(\frac{d\phi_0(y)}{dy} \right)^{-1} \frac{1}{-y(1+y)} dy \\
 &= \frac{1}{2\sqrt{l(l+1)}} \int_{y_{0a}}^{y_{0b}} \frac{-2l(1+l)y^2 + [-2l(1+l) + \frac{(2l+1)\beta^2}{l+1}]y - \frac{[(l+1)^2 - \beta^2][l(l+1) - \beta^2]}{2(l+1)^2}}{y(y+1)\sqrt{(y-y_{0b})(y_{0a}-y)}} dy = [l - \sqrt{l(l+1)}] \pi.
 \end{aligned}
 \tag{23}$$

Nex we calculate the left side of (7) for the $\tilde{V}_{eff}(y)$. For simplicity, we write $k(y)$ as follows

$$k(y) = \sqrt{-l(1+l)y^2 + [(\beta^2 - l(1+l)]y - \epsilon^2} = \sqrt{l(1+l)}\sqrt{(y_a - y)(y - y_b)},
 \tag{24}$$

and so the two turning points are:

$$\begin{aligned}
 y_a &= \frac{e^{-x_a}}{1 - e^{-x_a}} = \frac{\beta^2 - l(l+1) + \sqrt{[\beta^2 - l(l+1)]^2 - 4l(l+1)\epsilon^2}}{2l(l+1)}, \\
 y_b &= \frac{e^{-x_b}}{1 - e^{-x_b}} = \frac{\beta^2 - l(l+1) - \sqrt{[\beta^2 - l(l+1)]^2 - 4l(l+1)\epsilon^2}}{2l(l+1)}.
 \end{aligned}
 \tag{25}$$

$$\begin{aligned}
 y_a &= \frac{e^{-x_a}}{1 - e^{-x_a}} = \frac{\beta_1^2 + \sqrt{\beta_1^4 - 4\epsilon^2\beta_2^2}}{2\beta_2^2}, \\
 y_b &= \frac{e^{-x_b}}{1 - e^{-x_b}} = \frac{\beta_1^2 - \sqrt{\beta_1^4 - 4\epsilon^2\beta_2^2}}{2\beta_2^2},
 \end{aligned}
 \tag{26}$$

determined by $k(y) = 0$ and $x_a < x_b$. Now, it is easy to

find

$$\int_{x_a}^{x_b} k(x) dx = \int_{y_a}^{y_b} k(y) \frac{1}{-y(1+y)} dy = -\pi(\sqrt{l(1+l)} + \sqrt{\epsilon^2 - \sqrt{\beta^2 + \epsilon^2}}) \quad (27)$$

Substituting (23) and (27) into (7) and further using (13), we finally obtain

$$\epsilon_{nl} = \pm \frac{\beta^2 - (1+l+n)^2}{2(1+l+n)}, \quad (28)$$

$$E = -\frac{\hbar^2}{2M} \left[\frac{MZe^2}{\hbar^2(n+l+1)} - \frac{(n+l+1)\delta}{2} \right]^2,$$

here $N = n + 1$ has been used and n is the number of nodes of the wave function $R(r)$. This result is in complete agreement with (32) and (33) of [10].

4. Eigenfunctions

After obtaining the energy eigenvalues, we can find the corresponding eigenfunctions. Although we can directly

solve the Schrödinger equation for the effective potential $\tilde{V}_{eff}(x)$, we have considered another way to solve this problem. By using the variable y defined before and using (12) and (13), The Riccati equation

$$\frac{d}{dx} \phi(x) = -k(x)^2 - \phi(x)^2. \quad (29)$$

takes the form

$$y(1+y)\phi'(y) + l(1+l)y(1+y) - \beta^2y + \epsilon^2 - \phi(y)^2 = 0. \quad (30)$$

Using the MATHEMATICA (Wolfram Research, Inc., Mathematica, Version 5.2, Champaign, IL) software package, we can find a physically acceptable solution for this align

$$\phi(y) = -\frac{y(1+y)}{{}_2F_1(1+l+\epsilon+\sqrt{\beta^2+\epsilon^2}, -l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 1+2\epsilon, -y)} \cdot \left[\frac{\epsilon+y\epsilon+y\sqrt{\beta^2+\epsilon^2}}{y(1+y)} {}_2F_1(1+l+\epsilon+\sqrt{\beta^2+\epsilon^2}, -l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 1+2\epsilon, -y) - \frac{(-l+\epsilon+\sqrt{\beta^2+\epsilon^2})(1+l+\epsilon+\sqrt{\beta^2+\epsilon^2})}{1+2\epsilon} {}_2F_1(2+l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 1-l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 2+2\epsilon, -y) \right], \quad (31)$$

where ${}_2F_1$ is the Gauss hypergeometric function. From $u(x) = \exp\left(\int^x \phi(x) dx\right) = \exp\left(-\int^y \frac{\phi(y)}{y(1+y)} dy\right)$, we finally obtain

$$u(y) = Cy^\epsilon(1+y)^{\sqrt{\beta^2+\epsilon^2}} {}_2F_1(1+l+\epsilon+\sqrt{\beta^2+\epsilon^2}, -l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 1+2\epsilon, -y), \quad (32)$$

where C is the normalization constant. (32) is formally different from one given by (36) of [10], but we can prove they are identical each other. For this purpose, we introduce a variable $z = e^{-x}$ used by [9], $u(y)$ becomes

$$u(z) = z^\epsilon(1-z)^{-\epsilon-\sqrt{\beta^2+\epsilon^2}} {}_2F_1(1+l+\epsilon+\sqrt{\beta^2+\epsilon^2}, 1+2\epsilon-(1+l+\epsilon-\sqrt{\beta^2+\epsilon^2}), 1+2\epsilon, \frac{z}{z-1}). \quad (33)$$

Now, using formula (<http://functions.wolfram.com/07.23.16.0002.01>)

$${}_2F_1\left(a, c-b, c, \frac{z}{z-1}\right) = (1-z)^a {}_2F_1(a, b, c, z), \quad z \notin (1, \infty), \quad (34)$$

Table 2. Energy eigenvalues of the Hulthén potential as a function of the screening parameter for 4p, 4d, 4f, 5p, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$.

State	δ	EQR [5, 6]	AIM [10]	SUSY [16]	Numerical [15]	schroedinger [17]
4p	0.025	0.0200000	0.0200000	0.0199480	0.0199489	0.0199489
	0.050	0.0112500	0.0112500	0.0110430	0.0110582	0.0110582
	0.075	0.0050000	0.0050000	0.0045385	0.0046219	0.0046219
	0.100	0.0012500	0.0012500	0.0004434	0.0007550	0.0007550
4d	0.025	0.0200000	0.0200000	0.0198460	0.0198462	0.0198462
	0.050	0.0112500	0.0112500	0.0106609	0.0106674	0.0106674
	0.075	0.0050000	0.0050000	0.0037916	0.0038345	0.0038345
4f	0.025	0.0200000	0.0200000	0.0196911	0.0196911	0.0196911
	0.050	0.0112500	0.0112500	0.0100618	0.0100620	0.0100620
	0.075	0.0050000	0.0050000	0.0025468	0.0025563	0.0025563
5p	0.025	0.0094531	0.0094531	0.0094011	0.0094036	0.0094035
	0.050	0.0028125	0.0028125	0.0026056	0.0026490	0.0026490
5d	0.025	0.0094531	0.0094531	0.0092977	0.0093037	0.0093037
	0.050	0.0028125	0.0028125	0.0022044	0.0023131	0.0023131
5f	0.025	0.0094531	0.0094531	0.0091507	0.0091521	0.0091521
	0.050	0.0028125	0.0028125	0.0017421	0.0017835	0.0017835
5g	0.025	0.0094531	0.0094531	0.0089465	0.0089465	0.0089465
	0.050	0.0028125	0.0028125	0.0010664	0.0010159	0.0010159
6p	0.025	0.0042014	0.0042014	0.0041493	0.0041548	0.0041548
6d	0.025	0.0042014	0.0042014	0.0040452	0.0040606	0.0040606
6f	0.025	0.0042014	0.0042014	0.0038901	0.0039168	0.0039168
6g	0.025	0.0042014	0.0042014	0.0036943	0.0037201	0.0037201

we can write $u(z)$ as follows

$$u(z) = z^\epsilon (1 - z)^{1+l} {}_2F_1(1 + l + \epsilon + \sqrt{\beta^2 + \epsilon^2}, \epsilon + l + 1 - \sqrt{\beta^2 + \epsilon^2}, 1 + 2\epsilon, z). \tag{35}$$

If we let $1 + l + \epsilon + \sqrt{\beta^2 + \epsilon^2} = -n$, we will get (28) again. So $u(z)$ finally takes the form

$$u(z) = N_{nl} z^{\epsilon_{nl}} (1 - z)^{1+l} {}_2F_1(-n, 2(\epsilon_{nl} + l + 1) + n, 1 + 2\epsilon_{nl}, z), \tag{36}$$

where N_{nl} is a new normalization constant. By using following formula (Paul Abbott private communication,2007)

$$\int_0^1 (1 - z)^{2(\delta+1)} z^{2\lambda-1} {}_2F_1(-n, 2(\delta + \lambda + 1) + n; 2\lambda + 1; z)^2 dz = \frac{(n + \delta + 1)n! \Gamma(n + 2\delta + 2) \Gamma(2\lambda) \Gamma(2\lambda + 1)}{(n + \delta + \lambda + 1) \Gamma(n + 2\lambda + 1) \Gamma(2(\delta + \lambda + 1) + n)},$$

$$\delta > -\frac{3}{2} \wedge \lambda > 0, \tag{37}$$

we obtain the analytical expression of normalization constant

$$N_{nl} = \sqrt{\frac{\delta(n + l + \epsilon_{nl} + 1) \Gamma(n + 2\epsilon_{nl} + 1) \Gamma(2(l + \epsilon_{nl} + 1) + n)}{(n + l + 1)n! \Gamma(n + 2l + 2) \Gamma(2\epsilon_{nl}) \Gamma(2\epsilon_{nl} + 1)}}. \tag{38}$$

(36) is just the total radial wave function given by (36) of [10], but (38) is new.

5. Numerical application

values of the screening parameter and $Z = 1$. In Tables 1

To test the accuracy of our method, we calculate the energy eigenvalues for any n and l quantum numbers, several

and 2, EQR results are compared with the results of AIM [10], SUSY [16], the numerical integration [15] methods and the *schrodinger* Mathematica package [17]. As can be seen from these results, the EQR values are in good agreement with those from the other methods for small δ values. For large δ values, the differences between techniques become more apparent. The differences are due to the $\tilde{V}_{eff}(r)$ potential, which has been used to approximate the $V_{eff}(r)$ potential.

6. Concluding remarks

By using the Exact Quantization Rule, we have shown an alternative method to obtain the energy eigenvalues and corresponding eigenfunctions of the Hulthén potential using an approximation to the centrifugal potential. The main results of this paper are the energy eigenvalues which are given by equation (28). Furthermore, we obtained normalized eigenfunctions given by (32) and (36). The advantage of the Exact Quantization Rule is that it gives the eigenvalues by calculating two integrations, and solving the resulting algebraic equation. The wave functions are easily obtained by solving the Riccati equation and further completing an integration. The method presented in this paper is a systematic one and it is very efficient and practical. It is worth extending this method to the solution of other interaction problems.

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