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Double-Alekseev inverse scattering method in the stationary axi-symmetric vacuum gravitation field equations

Research Article

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Abstract: We present a new improvement to the Alekseev inverse scattering method. This improved inverse scattering method is extended to a double form, followed by the generation of some new solutions of the double-complex Kinnersley equations. As the double-complex function method contains the Kramer-Neugebauer substitution and analytic continuation, a pair of real gravitation soliton solutions of the Einstein's field equations can be obtained from a double N-soliton solution. In the case of the flat Minkowski space background solution, the general formulas of the new solutions are presented.

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1. Introduction

In theories of gravitation, it is generally accepted that Einstein's theory of general relativity is the most successful. Einstein's field equations are at the core of the general relativity and their exact solutions have played very important roles in the discussion of this theory. However, it should also be noted that the Einstein's field equations are highly non-linear differential equations, making it very difficult to construct and investigate families of exact solutions. Fortunately, if restricted to the class of two-dimensional field configurations, Einstein's field equations have remarkable properties such as their integrability and internal symmetry. Thanks to these properties, these space-time symmetrical Einstein's equations have proven to be a broad arena for diverse mathematical methods, such as the inverse scattering method [2–8], group-theory approach [9, 10], Backlund transformation [11–14] and so on, with each method finding applicability in different contexts.

The inverse scattering technique of V.A. Belinski and V.E. Zakharov (BZ) is well known [2, 3]. It is a solution-generating procedure for producing exact vacuum solu-

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tions. In this method, the authors put non-linear equations, which can be equivalent to the Ernst equations [1], in line with two-dimensional linear equations with a spectral parameter, thereby obtaining exact vacuum soliton solutions by the algebraic operation, having started from some initial seed solutions. The alternative inverse scattering method of Alekseev, which similarly generates solutions of the Einstein-Maxwell equations, is developed by the author in [5]. For specific examples see [6–8]. If the electromagnetic field vanishes, these electromagnetic solitons become purely gravitational and may be equivalent to those obtained by the BZ inverse scattering method. However, it is based on another different complex expression (Kinnersley equations) in [9, 10] of the field equations, therefore the formulations of this technique are different from the formulations of BZ. Further, it should also be denoted that this method has many advantages, e.g. the pole situations are constants and can be naturally extended to higher-dimensions. But unlike the BZ inverse scattering method, this technique has not been discussed widely. In this article we will further discuss the Alekseev inverse scattering method in the case of stationary axi-symmetric vacuum gravitation fields, in particular, we will consider extending this method to the double [15] form in order to obtain more solutions of Einstein's field equations.

The double-complex function method in [15], which organically combines the ordinary complex with the hyperbolic complex [16] function theories, has been effectively used to study some mathematical physics problems. In these studies, the double-complex function method and its extended version [17, 18] have been extensively used in Einstein's gravity theories and string theories [19–22] as powerful tools for finding new symmetries and generating new solutions of the associated field equations. It is known that the double-complex function method contains both the Kramer-Neugebauer (NK) substitution [23] $(f,\psi)\rightarrow(\rho f^{-1}, i\psi)$ and analytic continuation, thus a pair of real solutions can be derived from a double-complex solution automatically.

This article is organized as follows: in Section 2, we consider an improvement of the restriction conditions of the soliton-generating technique. In Section 3, some related concepts and notations of the double-complex numbers are briefly reviewed, then we present a new approach to generate solutions of Einstein's vacuum field equations, which we call the "double-Alekseev inverse scattering method". In Section 4, we apply this method to existing general formulas of new solutions on a background solution of flat Minkowski space.

2. Alekseev inverse scattering method and improvement

2.1. Alekseev inverse scattering method

Let us write the metric of a stationary axi-symmetric vacuum gravitation field in the following form

$$ds^2 = -\ell\delta_{\mu\nu}dx^\mu dx^\nu + g_{ab}dx^a dx^b, \quad \delta_{\mu\nu} = \left\{ \begin{array}{l} 0(\mu\neq\nu) \\ 1(\mu=\nu) \end{array} \right. . \tag{1}$$

where indices $\mu,\nu = 0,1; a,b = 2,3$ and the conformal factor ℓ ($\ell > 0$) and the metric g_{ab} are functions of the coordinates x^μ. The coordinates are$(x^0,x^1,x^2,x^3) = (\rho,z,t,\phi)$, respectively. In this article we only consider the solutions of g_{ab}. The Einstein field equations about $g = \{g_{ab}\}$ can be written as [2, 3]

$$\partial_\rho(\rho g^{-1}\partial_\rho g) + \partial_z(\rho g^{-1}\partial_z g) = 0, \tag{2}$$

where g is a 2×2 real symmetric matrix, and must satisfy the condition $\det g = -\rho^2$. For g we use a parametrization,

$$g = \left(\begin{array}{cc} f & -f\omega \\ -f\omega & f\omega^2 - \rho^2 f^{-1} \end{array} \right), (f>0), \tag{3}$$

where f,ω are functions of x^μ. Next we can obtain the Kinnersley equations in [9, 10] by using (2)

$$\partial_\mu H = -i\rho^{-1}\varepsilon_\mu^\nu h\partial_\nu H \quad \varepsilon_\mu^\nu = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \tag{4}$$

where the 2×2 matrix $H = \left\{H_a^b\right\}$ is the Ernst-like potential and $h = ReH$. Among various gauge equivalent linear systems with coordinate dependent spectral parameters, we choose the Kinnersley-like linear system whose appropriately normalized fundamental solutions seem to possess the simplest general analytical structure on the spectral plane

$$\partial_\mu\psi = \Lambda_\mu^\nu U_\nu\psi \quad \Lambda_\mu^\nu = \frac{1}{2i}\frac{(w-z)\delta_\mu^\nu - \rho\varepsilon_\mu^\nu}{(w-z)^2+\rho^2}, \tag{5}$$

where $\psi = \psi(x^\mu, w)$ is a 2×2 matrix wave function and w is an arbitrary ordinary complex spectral parameter. The 2×2 matrices $U_\mu = \partial_\mu H$ are related to the metric components g_{ab} through the additional conditions in reference [5]

$$-2\partial_\mu(\sigma h + iz\sigma) = (U_\mu^+\sigma - \sigma U_\mu), \tag{6}$$

$$-\sigma h U_\mu = i\rho\varepsilon_\mu^\nu\sigma U_\nu, \tag{7}$$

$$trU_\mu = 2i\partial_\mu z, \quad (8)$$

with $h = \{h_a^b\} = -g\sigma$ and $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. At the same time, the wave function ψ in (5) should satisfy the first integral condition, $\psi^+ W \psi = V(w)$, where $\psi^+(x^\mu, w) \equiv [\psi(x^\mu, \bar{w})]^+$, $V(w)$ is an arbitrary 2×2 Hermitian matrix depending on w only, and the 2×2 matrix function W is linear in w, which satisfies

$$W \equiv -\sigma h + i(w - z)\sigma. \quad (9)$$

If supplied with seed metric g_0 and an associated matrix $\psi_0(x^\mu, w)$, a new solution for the linear pair is obtained from the transformation $\psi(x^\mu, w) = \chi(x^\mu, w)\psi_0(x^\mu, w)$, where the matrix χ and its inverse are assumed to have the following form,

$$\chi = \mathrm{I} + \sum_{k=1}^{N} \frac{R_k}{w - w_k}, \quad \chi^{-1} = \mathrm{I} + \sum_{k=1}^{N} \frac{S_k}{w - \tilde{w}_k}. \quad (10)$$

therefore, for any choice of the background solution the generating N-soliton solutions can be expressed in terms of the background $\psi_0(x^\mu, w)$ and a set of constants in reference [8], that are integrable.

2.2. Improvement of the formulas

As stated above, the whole integrable problem of Einstein's vacuum field equations for two-dimensional configurations is equivalent to the determination of matrices $\psi(x^\mu, w)$, $U_\mu(x^\mu, w)$ and $W(x^\mu, w)$. These matrices must satisfy the restriction conditions (6) - (8) and the first integral condition. But it can be seen that Eqs. (6) - (8) do not seem very attractive, so we consider making some improvements with the additional requirements. Let us define new space-time coordinates variables ξ and η

$$\xi = z + i\rho \quad \eta = z - i\rho, \quad (11)$$

then the Kinnersley equations (4) can be written in the new coordinates as follows

$$\sigma h \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) H = -\rho\sigma \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) H, \quad (12)$$

$$\sigma h \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) H = -\rho\sigma \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) H. \quad (13)$$

Then, considering the matrix W in variables ξ and η, we obtain a new expression

$$W = -\sigma h + i\left[w - \frac{1}{2}(\xi + \eta)\right]\sigma. \quad (14)$$

So, combining (12) with (13) and considering (14), we have

$$W(w = \xi)\partial_\xi H = 0, \quad (15)$$

$$W(w = \eta)\partial_\eta H = 0, \quad (16)$$

on the locations $w = \xi$ and $w = \eta$, respectively. It is obvious that the Kinnersley equations can be reformulated like (15) and (16), identically. Finally, we substitute the relation of U_μ and H into (15) and (16) to obtain

$$W(w = \xi)U_\xi = 0, \quad (17)$$

$$W(w = \eta)U_\eta = 0. \quad (18)$$

It is obvious that if the new solutions U_ξ, U_η, W, which we obtained by the inverse scattering method satisfy Eqs. (17) and (18), then they become the solutions of the Kinnersley equations to a certainty. Therefore, we find that the new expressions with the additional requirements are extremely regular and thus avoid the complexity of the original expressions.

3. Double-Alekseev inverse scattering method

3.1. Mathematics preparation

The double-complex function method in [15] is a mathematical physics method which combines the ordinary complex numbers with hyperbolic complex numbers [16] by an "analysis link" J and employs both complex numbers simultaneously. Now, we have to collect some essential symbols. Let J denotes the double imaginary units, i.e.$J = i(i^2 = -1)$, or $J = \varepsilon(\varepsilon^2 = +1, \varepsilon \neq \pm 1)$. If a series $\sum_{n=0}^{\infty} |a_n|, a_n \in R$ (real number field) is convergent, then

$$a(J) = \sum_{n=0}^{\infty} a_n J^{2n},$$

is called a double real number, which corresponds to a pair $(a_C, a_H)$of ordinary real numbers, where $a_C \equiv a(J = i)$, $a_H \equiv a(J = \varepsilon)$. If all of the double real numbers with ordinary addition and multiplication constitute a field, we call it a double real number field and denote it by $R(J)$. When $a(J)$ and $b(J)$ are both double real numbers, then

$$c(J) = a(J) + Jb(J),$$

is called a double complex number, which corresponds to a pair (c_C, c_H), where $c_C \equiv c(J=i) = a_C + ib_C$ is an ordinary complex number, and $c_H \equiv c(J=\varepsilon) = a_H + \varepsilon b_H$ is a hyperbolic complex number. From the above definitions, we can see that the double imaginary unit J takes the role of an analysis link between c_C and c_H. All double complex numbers with usual addition and multiplication constitute a commutative ring, which is denoted by $C(J)$. The double complex conjugation of a double complex number $c(J)$ is defined by $\overline{c(J)} \equiv a(J) - Jb(J)$. This implies that $\overline{J} = -J$.

3.2. The double inverse scattering method

Now, we extend the Alekseev inverse scattering method to the double form. From Sec. 2 we know that the line element of the metric of a stationary axi-symmetric vacuum gravitation field can be written as (1), and $g\sigma g = \det g\sigma$ is satisfied automatically. The field equation similar to reference [1] is

$$f\nabla^2 f = \nabla f\cdot\nabla f - \rho^{-2} f^4 \nabla\omega\cdot\nabla\omega, \tag{19}$$

$$\nabla\cdot(\rho^{-1} f^2 \nabla\omega) = 0, \tag{20}$$

where ∇ denotes the two-dimensional divergence operator with $\nabla = (\partial_\rho, \partial_z)$, $\nabla^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$. Transformations T and V are defined by

$$T : f \to T(f) = \rho f^{-1},$$

$$V : (f, \phi) = V_f(\phi) = \omega, \tag{21}$$

$$\omega = V_f(\phi) = \int \rho f^{-2}(\partial_z\phi d\rho - \partial_\rho\phi dz),$$

where ϕ is real function of ρ and z.
Thus, if we let $(f, \omega) = (F_C, \Omega_C)$, Eqs. (19) and (20) are changed into

$$F_C\nabla^2 F_C = \nabla F_C\cdot\nabla F_C - \rho^{-2} F_C^4 \nabla\Omega_C\cdot\nabla\Omega_C, \tag{22}$$

$$\nabla\cdot(\rho^{-1} F_C^2 \nabla\Omega_C) = 0, \tag{23}$$

with the matrix $g = g_C = \begin{pmatrix} F_C & -F_C\Omega_C \\ -F_C\Omega_C & F_C\Omega_C^2 - \rho^2 F_C^{-1} \end{pmatrix}$, and under the transformation $(f,\omega) \to (F_H, \Omega_H) = (T^{-1}(f), V_f^{-1}(\omega)) = (\rho f^{-1}, \phi)$, Eqs. (19) and (20) are changed into

$$F_H\nabla^2 F_H = \nabla F_H\cdot\nabla F_H + \rho^{-2} F_H^4 \nabla\Omega_H\cdot\nabla\Omega_H, \tag{24}$$

$$\nabla\cdot(\rho^{-1} F_H^2 \nabla\Omega_H) = 0, \tag{25}$$

with the matrix $g = g_H = \begin{pmatrix} F_H & -F_H\Omega_H \\ -F_H\Omega_H & F_H\Omega_H^2 + \rho^2 F_H^{-1} \end{pmatrix}$.
Comparing the matrices g_C with g_H, we can introduce the double imaginary unit J, and obtain a double real matrix

$$g(J) = \begin{pmatrix} F(J) & -F(J)\Omega(J) \\ -F(J)\Omega(J) & F(J)\Omega^2(J) + J^2\rho^2 F^{-1}(J) \end{pmatrix}, \tag{26}$$

which satisfies the double vacuum field equations

$$\partial_\rho[\rho g^{-1}(J)\partial_\rho g(J)] + \partial_z[\rho g^{-1}(J)\partial_z g(J)] = 0. \tag{27}$$

And then, with the transformation $h(J) = -g(J)\sigma$, we can obtain another form of the field equations as follows

$$\partial_\rho[\rho^{-1} h(J)\partial_\rho h(J)] + \partial_z[\rho^{-1} h(J)\partial_z h(J)] = 0. \tag{28}$$

It is obvious that our field equations imply the existence of a set of potentials $\varphi(J)$

$$\partial_z\varphi(J) = \rho^{-1} h(J)\partial_\rho h(J), \tag{29}$$

$$-\partial_\rho\varphi(J) = \rho^{-1} h(J)\partial_z h(J). \tag{30}$$

Eqs. (29) and (30) may be inverted and solved for $\partial_\mu h(J)$

$$\partial_z h(J) = J^2\rho^{-1} h(J)\partial_\rho\varphi(J), \tag{31}$$

$$\partial_\rho h(J) = -J^2\rho^{-1} h(J)\partial_z\varphi(J). \tag{32}$$

In terms of the double complex function $H(J) = h(J) + J\varphi(J)$, Eqs. (29) - (32) may be combined into a single double complex equations

$$\partial_\mu H(J) = -J\rho^{-1}\varepsilon_\mu^\nu h(J)\partial_\nu H(J), \tag{33}$$

called "the double-complex Kinnersley equations".
Of course, there is a problem of how to generate new solutions of these double complex equations. Therefore, the ordinary inverse scattering method should be extended to the following double form. Firstly, we can obtain double non-linear system equations from the double-complex Kinnersley equations as follows

$$\delta^{\mu\nu}\partial_\mu U_\nu(J) - \frac{J^3}{2\rho}\varepsilon^{\mu\nu} U_\mu(J) U_\nu(J) = 0, \tag{34}$$

where $U_\mu(J) = \partial_\mu H(J)$ are 2×2 double complex matrix functions which depend on x^μ. Taking Eqs. (34) as the consistency condition, we look for the required double linear system in a especial form in the coordinates ξ and η

$$-2J^3(w-\xi)\partial_\xi\psi(J) = U_\xi(J)\psi(J) \quad (35)$$

$$-2J^3(w-\eta)\partial_\eta\psi(J) = U_\eta(J)\psi(J) \quad (36)$$

where $U_\xi(J) = U_\mu(J)\left(\frac{\partial x^\mu}{\partial \xi}\right)$, $U_\eta(J) = U_\mu(J)\left(\frac{\partial x^\mu}{\partial \eta}\right)$, and the wave function $\psi(J) = \psi(\xi, \eta, w; J)$ is a 2×2 double complex matrix, w is an ordinary complex spectral parameter. Because (34) are only consequences of the Kinnersley equations, but are not equivalent to (33), we should have certain additional restrictions on the solutions $U_\mu(J)$, namely

$$W(w=\xi; J)\, U_\xi(J) = 0, \quad (37)$$

$$W(w=\eta; J)\, U_\eta(J) = 0, \quad (38)$$

where the double complex 2×2 Hermitian matrix $W(J)$ satisfies $W(J) \equiv -\sigma h(J) - J^3(w-z)\sigma$. At the same time, the wave function $\psi(J)$in (35) and (36) should satisfy the first integral condition

$$\psi^+(J)W(J)\psi(J) = V(w;J), \quad V(w;J) = V^+(w;J), \quad (39)$$

where $V(w;J)$ is arbitrary and coordinates independent.
It can be seen that the whole problem of the double nonlinear system corresponds to research and involves calculating the problem of the double linear system with spectral parameters (35) and (36) under the additional conditions (37) and (38) and the first integral condition (39). In the next section we will give the general algorithm for the generation of double-soliton solutions.

3.3. Generation of double-soliton solutions

After it became possible to reformulate the problem of integrating the original double non-linear system in terms of an over-determined double linear system with a complex parameter, the double-soliton solutions could be constructed most simply by a method analogous to that used in reference [8]. To do this we make a change of unknown variables

$$\psi(J) = \chi(J)\psi_0(J), \quad (40)$$

in the double linear system (35-36), where $\chi(\xi, \eta, w; J)$ is a new unknown 2×2 matrix function, and $\psi_0(\xi, \eta, w; J)$ is a known quantity defined as a certain fundamental solution of (35-36), the matrices $U_{(0)\mu}(J)$ in (35-36) can be computed from a certain previously chosen exact solution $g_0(J)$. Considering (35-36) and (40), the relations between new solutions $U_\xi(J), U_\eta(J)$ with the scattering matrix $\chi(J)$ and its inverse $\chi^{-1}(J)$ can be written in such a way as to have no singularities for $w=\xi$ and $w=\eta$

$$U_\xi(J) = \chi(w=\xi;J)U_{(0)\xi}(J)\chi^{-1}(w=\xi;J), \quad (41)$$

$$U_\eta(J) = \chi(w=\eta;J)U_{(0)\eta}(J)\chi^{-1}(w=\eta;J). \quad (42)$$

It is obvious that to construct N-soliton solutions, we assume that on the plane of the complex parameter w the matrices $\chi(J)$ and $\chi^{-1}(J)$ have the following structures, and it will become the identity matrix for $w\to\infty$

$$\chi(J) = \mathrm{I} + \sum_{k=1}^{N}\frac{R_k(J)}{w-w_k}, \quad \chi^{-1}(J) = \mathrm{I} + \sum_{k=1}^{N}\frac{S_k(J)}{w-\tilde{w}_k}, \quad (43)$$

$$R_k(J) = n_k(J)\otimes m_k(J), \quad S_k(J) = p_k(J)\otimes q_k(J), \quad (44)$$

here the index "k" runs from 1 to N, The equations (44) take into account the fact that both $R_k(J)$ and $S_k(J)$ must be matrices with a vanishing determinant. Finally, the vectors $n_k(J), m_k(J), p_k(J)$ and $q_k(J)$ are given in terms of arbitrary double complex constant vectors $k_k(J)$ and $l_k(J)$ by

$$q_k(J) = -\sum_{l=1}^{N}\Gamma_{lk}^{-1}(J)m_l(J), \quad n_k(J) = \sum_{l=1}^{N}\Gamma_{kl}^{-1}(J)p_l(J),$$

$$\Gamma_{kl}(J) = \frac{[m_l(J)\cdot p_k(J)]}{w_l-\tilde{w}_k},$$

$$m_k(J) = k_k(J)\cdot\psi_0^{-1}(w_k;J), \quad p_k(J) = \psi_0(\tilde{w}_k;J)\cdot l_k(J). \quad (45)$$

By (39) and (40) we can write the form $\chi^+(J)W(J) = W_0(J)\chi^{-1}(J)$, which leads to relations among the arbitrary constants and allows us to express all the w_k in terms of $\tilde{w}_k$, and all the $l_k(J)$ in terms of $k_k(J)$

$$\tilde{w}_k = \bar{w}_k, \quad l_k(J) = V_0^{-1}(J)k_k^+(J). \quad (46)$$

Following the above, we can state that new solutions obtained by the double inverse scattering method satisfy the additional restrictions (28), so they are solutions of the original field equations.
Finally, we present formulas enabling us to explicitly express the double N-soliton solutions

$$U_\mu(J) = U_{(0)\mu}(J) - 2J^3\partial_\mu R(J), \quad (47)$$

where the elements of the matrix $R(J)$ are computed from the formula

$$R(J) = \sum_{k=1}^{N}R_k(J) = \sum_{k=1}^{N}n_k(J)\otimes m_k(J)$$
$$= \sum_{k,l=1}^{N}\Gamma_{kl}^{-1}(J)p_l(J)\otimes m_k(J), \quad (48)$$

with (44)-(46) taken into account.

4. Application: solutions generated on a background of flat minkowski space

In the preceding section we described the double-Alekseev inverse scattering method for generating soliton solutions of Einstein's field equations. To clarify the physical content of the various procedures for generating solutions, it is useful to consider the metric of flat Minkowski space as the original solution. Below we apply the double-Alekseev inverse scattering method to Einstein's field equations on a background of flat Minkowski space and give the general formulas of the new solutions. The line element of the metric can be written as

$$-ds^2 = -dt^2 + \rho^2 d\phi^2 + d\rho^2 + dz^2. \qquad (49)$$

We choose the seed solution $g(J)$ [15]

$$g_0(J) = \begin{pmatrix} 1 & 0 \\ 0 & J^2\rho^2 \end{pmatrix}. \qquad (50)$$

Next, we present general formulas for computing N-soliton solutions on a background of the flat metric (50). For this background we can obtain the original solutions of $U_\mu(J)$ and $W_\mu(J)$, respectively

$$U_{(0)\mu}(J) = -J^2 \begin{pmatrix} 2J & 0 \\ -2\rho & 0 \end{pmatrix},$$
$$W_0(J) = -J^2 \begin{pmatrix} \rho^2 & J(w-z) \\ -J(w-z) & J^2 \end{pmatrix}. \qquad (51)$$

and we can testify that the matrices $U_{(0)\mu}(J)$, $W_0(J)$ satisfy the additional restrictions (37) and (38).

Certainly, we can calculate the wave function and its inverse by the double linear system and the first integral condition. Furthermore, without loss of generality, we choose $V_0(J) = \begin{pmatrix} -J^2 & 0 \\ 0 & -1 \end{pmatrix}$ in Eq. (39)

$$\psi_0(J) = \begin{pmatrix} \frac{1}{Q(J)} & 0 \\ \frac{w-z}{JQ(J)} & -J^2 \end{pmatrix}, \quad \psi_0^{-1}(J) = \begin{pmatrix} Q(J) & 0 \\ J(w-z) & -J^2 \end{pmatrix}. \qquad (52)$$

where $Q(J) = -J^2\sqrt{(w-z)^2+\rho^2}$ is a double auxiliary scalar function.

Thus, the general formulas in the preceding section yield expressions of the vectors m_k and p_k, when we choose $k_k(J) = (\ c_k(J) \quad d_k(J)\ )$ with the index "k" runs from 1 to N

$$m_k(J) = (\ Q_k(J)c_k(J) + J\,(w_k - z)\,d_k(J) \quad -J^2 d_k(J)\ ), \qquad (53)$$
$$p_k(J) = \begin{pmatrix} \frac{-J^2 c_k(J)}{\bar{Q}_k(J)} \\ \frac{-J(\bar{w}_k - z)\bar{c}_k(J)}{\bar{Q}_k(J)} + J^2 \bar{d}_k(J) \end{pmatrix}, \qquad (54)$$

where the quantities $c_k(J)$ and $d_k(J)$ are arbitrary double complex constants. The elements of the 2×2 matrix $(\Gamma_{kl}(J))$ have the form

$$\Gamma_{kl}(J) = \frac{1}{w_l - \bar{w}_k}\left\{\frac{-J^2 Q_l(J)c_l(J)\bar{c}_k(J)}{\bar{Q}_k(J)} - \frac{J^3(w_l - \bar{w}_k)\bar{c}_k(J)d_l(J)}{\bar{Q}_k(J)} - d_l(J)\bar{d}_k(J)\right\}. \qquad (55)$$

Computing the components of $R(J)$ from (48), with (42) taken into account, the components obtain the form

$$R_{11} = \sum_{k,l=1}^{N} \Gamma_{kl}^{-1}(J)\frac{-J^2 Q_k(J)\bar{c}_l(J)c_k(J) - J^3(w_k - z)\bar{c}_l(J)d_k(J)}{\bar{Q}_l(J)},$$

$$R_{12} = \sum_{k,l=1}^{N} \Gamma_{kl}^{-1}(J)\frac{\bar{c}_l(J)d_k(J)}{\bar{Q}_l(J)},$$

$$R_{21} = \sum_{k,l=1}^{N} \Gamma_{kl}^{-1}(J)\frac{-J(\bar{w}_l - z)Q_k(J)\bar{c}_l(J)c_k(J) + J^2 Q_k(J)\bar{Q}_l(J)c_k(J)\bar{d}_l(J)}{\bar{Q}_l(J)} - \frac{J^2(w_k - z)(\bar{w}_l - z)\bar{c}_l(J)d_k(J) - J^3(w_k - z)\bar{Q}_l(J)\bar{d}_l(J)d_k(J)}{\bar{Q}_l(J)},$$

$$R_{22} = \sum_{k,l=1}^{N} \Gamma_{kl}^{-1}(J)\frac{J^3(\bar{w}_l - z)\bar{c}_l(J)d_k(J) - \bar{Q}_l(J)\bar{d}_l(J)d_k(J)}{\bar{Q}_l(J)}. \qquad (56)$$

Finally, we can calculate the general formulas of the components of the double metric $g(J)$ as follows

$$g_{11} = 1 + 2ImR_{12}, \quad g_{12} = -2ImR_{11},$$

$$g_{21} = 2ImR_{22}, \quad g_{22} = J^2\rho^2 - 2ImR_{21}. \tag{57}$$

It is obvious that when the parameters in matrix $R(J)$ are chosen appropriately, the new double-soliton solutions $g(J)$ can be generated by the double-Alekseev inverse scattering method, namely, $g(J) = g_C$ when $J = i$, and $g(J) = g_H$ when $J = \varepsilon$, respectively. We can simply call g_H the dual solution of g_C. Then by the T, V transformations, we can obtain two sequences of vacuum gravitation soliton solutions, one of which is a sequence of new solutions which cannot be solely generated by the ordinary complex function method.

5. Conclusion

In this article, we have improved the additional conditions of the Alekseev inverse scattering method and found new expressions that are simpler and more regular. We combine the double-complex function method with the Alekseev inverse scattering method and present the double-Alekseev inverse scattering method. When applying this method for generating soliton solutions of vacuum Einstein's equations on the background of flat Minkowski space, we can obtain a pair of gravitation real solutions from a double-soliton solution. This new method is useful to investigate the families of exact solutions of Einstein's field equations. Likewise, it should be noted that the double-Alekseev inverse scattering method is a potential candidate to consider when solving Einstein-Maxwell field as well as the string theory model.

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