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Effects of noise on periodic orbits of the logistic map

Research Article

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Abstract: Noise can induce an inverse period-doubling transition and chaos. The effects of noise on each periodic orbit of three different period sequences are investigated for the logistic map. It is found that the dynamical behavior of each orbit, induced by an uncorrelated Gaussian white noise, is different in the mergence transition. For an orbit of the period-six sequence, the maximum of the probability density in the presence of noise is greater than that in the absence of noise. It is also found that, under the same intensity of noise, the effects of uncorrelated Gaussian white noise and exponentially correlated colored (Gaussian) noise on the period-four sequence are different.

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Keywords: logistic map • noise • periodic orbits • probability density

1. Introduction

Noise-affected chaotic dynamics have been one of the most important and interesting topics in nonlinear dynamical systems. The logistic map is a specific and very simple dynamical system that exhibits a complex dynamical behavior. Up to now, the effect of noise on the logistic map have been extensively studied from several points of view, such as chaotic dynamics controlling [1, 2], coherence resonance [3], the effect of noise on the bifurcation behavior [4] and estimation of Lyapunov exponents [5], noise-induced order [6, 7] and noise-induced chaos [8–12]. When J. P. Crutchfield *et al.* [9] investigated the effects of fluctuations on the period-doubling bifurcation to chaos, they observed that noise can induce an inverse period-doubling cascade in the logistic map that is the period of sixteen merged into eight and the period of eight merged into four. G. Mayer-Kress and H. Haken [13] reported that a sequence of mergence transitions occurs at the critical parameter if the logistic system is perturbed by external noise. When the system itself is periodic, J. Perez and C. Jeffries [14] have observed this phenomenon in a nonlinear LRC oscillator.

However, few papers are devoted to investigate in detail how noise can affect each periodic orbit that belongs to the periodic sequence in the logistic map in the process of the mergence transitions. We emphasize on finding how noise can affect the changes of probability density of each periodic orbit. This study may be enlightening in order to further understand some aspects of the evolution of chaotic systems in the presence of noise.

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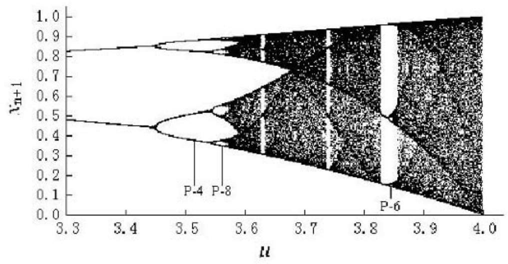

Figure 1. The bifurcation diagram for the Logistic map of Eq. (1). 150 iterations plotted after an initial 199850 iterations for each increment in the bifurcation parameter. The parameter was incremented 1000 times in the interval [3.3, 4]. For the sake of clarity and resolution, only the bifurcation diagram for μ in [3.3, 4] is shown.

2. The logistic map

We consider the logistic map. Its dynamical equation can be written as

$$x_{n+1} = \mu x_n(1 - x_n), 0 < x_n < 1, \tag{1}$$

where μ is the bifurcation parameter and n is a natural number. The equation is very simple, however, it shows very complex behavior (Fig. 1). In order to exhibit the details of the diagram, the bifurcation parameter μ starts from 3.3. As we can see from Fig. 1, it goes through a period-doubling cascade to chaos as μ increases; this is known as the period doubling route to chaos. In the chaotic region there are some interesting windows with periodic orbits of powers other than 2. Each window has a different width. The widest window is a three-period window followed by period 6 and 5 windows. Another interesting feature of this diagram is the high degree of self-similarity, where structure on one length scale in Fig. 1 is repeated on successively smaller scales.

3. The influence of the uncorrelated Gaussian white noise on each periodic orbit

We will investigate the effect of changes in the intensity of noise on the behaviors of Eq. (1). The model is described as follows:

$$x_{n+1} = \mu x_n(1 - x_n) + P_n. \tag{2}$$

Where the P_n denotes uncorrelated Gaussian white noise with zero mean and correlation function: $\langle P_m P_n \rangle = 2D\delta_{m,n}$ for $m, n = 1, 2, 3 \cdots$. $\langle \cdots \rangle$ denotes an ensemble average over the distribution of noise. δ is the Dirac delta function and D is the intensity of the noise. The bifurcation diagram induced by the noise can be found in Ref. 9 and 11. Here, we will concentrate our investigation on the behavior of the probability density of the periodic orbits induced by noise at the bifurcation parameter values $\mu = 3.5$, 3.55, and 3.846. Without noise, the logistic map at these parameter values is periodic with periods 4, 8, and 6 respectively. It should be mentioned that the logistic map belongs to the main 2^n cascade for $\mu = 3.5$ and 3.55 and the 3×2^n cascade for 3.846.

Fig. 2 shows the heights of the probability densities of the orbits as a function of x_{n+1} for different values of D=0, 0.0001, 0.004 and 0.01. When D=0, namely no noise, the probability density is a four-delta function. Their heights are equal to 0.25 and their positions are 0.384, 0.502, 0.827 and 0.874 respectively. As the noise intensity D is increased, the peaks broaden and eventually the period of four merges into a period of two. Thus, the noise induces the logistic map to change from a four-period sequence to a two-period sequence. In other words, noise can induce merging transition in the logistic map. Moreover, we also find that the amplitudes of the probability densities for each orbit in Fig. 2(b) and Fig. 2(c) are different, which shows that the effect of noise on each orbit is different. In order to further understand how noise affects the height of the probability densities for each orbit, we choose a series of different noise intensities and study the dynamical behavior of each orbit. Fig. 3 shows the heights of the probability densities as a function of the noise intensities D for four orbits. From Fig. 3, when D is very small, the heights of four orbits tend to a constant (0.25). As D is increased, the heights of the probability densities of the four orbits are monotonously decreased at different rates. For the orbits of 0.874 (as diamonds $\diamond$) and 0.827 (as up triangles $\triangle$), they start to decline from 0.25 when D becomes greater than 1×10^{-5}; the orbit of 0.827 decreases quicker than the orbit of 0.874. The heights of the orbits of 0.384 (as open circles $\circ$) and 0.502 (as stars $*$) start to decrease when D is more than 1×10^{-7} and the orbit of 0.502 presents a fluctuation during the decline in amplitude. So it can be concluded that the orbit of 0.874 in the period-four sequence is more stable than the others.

For $\mu = 3.55$, as D is increased, we also find that the period of eight merges into four and then the period of four merges into two [Fig. 4(a)-(d)]. The heights of probability densities of each orbit also decrease monotonously (Fig. 5). The decrease of the orbit of 0.888 (as crosses $+$) is slower than the others with increasing D. Comparing with Fig. 3, the orbits of 0.505 (as open left triangles $\triangleleft$) and 0.542 (as stars $*$) started to decline from 0.125 when D

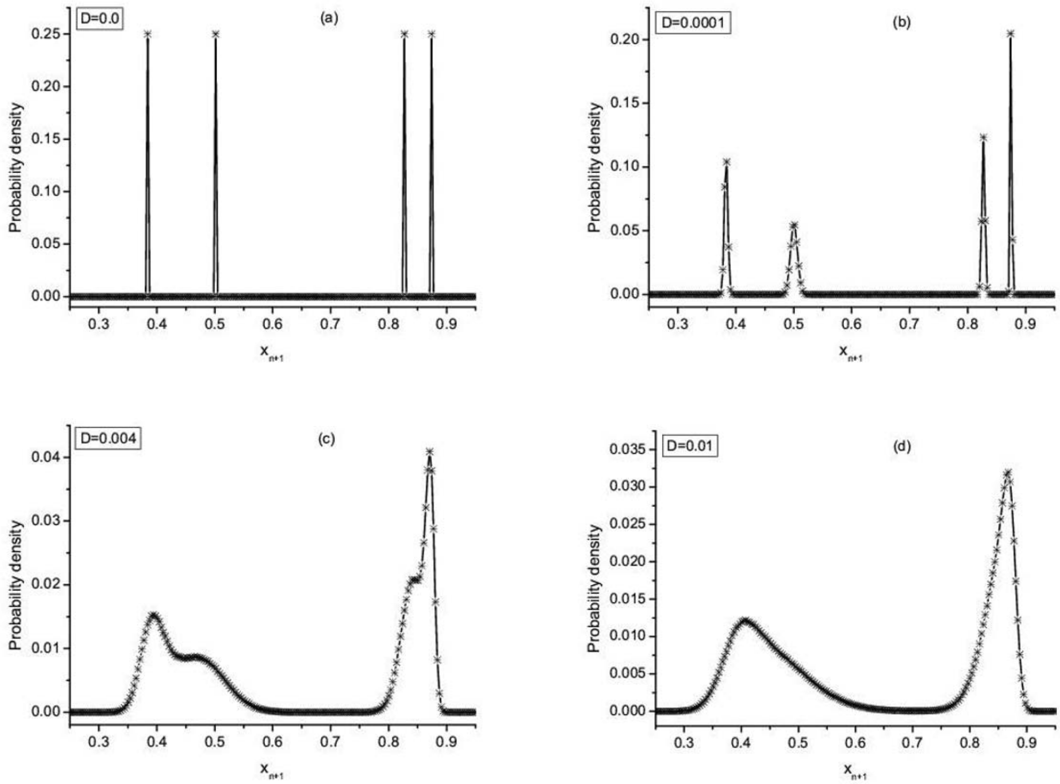

Figure 2. Plot of the normalized probability density, at the bifurcation parameter value $\mu = 3.5$, for different noise intensities D: (a) $D = 0.0$, (b) $D = 0.0001$, (c) $D = 0.004$, and (d) $D = 0.01$. 4×10^6 iterations, after an initial 6×10^6 iterations, were partitioned into 300 bins.

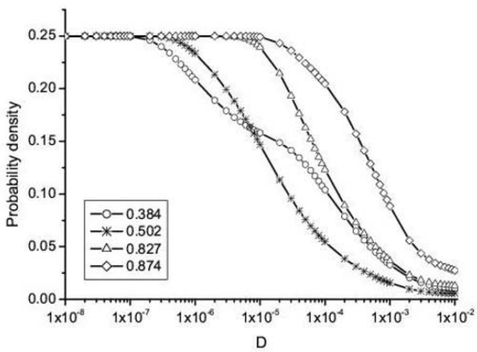

Figure 3. Plot of the normalized probability densities at the bifurcation parameter value $\mu = 3.5$ for $x_{n+1} = 0.384$ (open circles ∘), 0.502(stars ∗), 0.827(open up triangles △) and 0.874(diamonds ⋄), 4×10^6 iterations are calculated after an initial 6×10^6 iterations for each D.

is roughly more than 1×10^{-8} . This indicates that the period-eight sequence is more sensitive to noise.

Since the logistic map for $\mu = 3.846$ belongs to the 3×2^n cascade in the chaotic region, we are more interested in its dynamical behavior induced by the noise. Fig. 6 shows that, as D is increased, the period of six is merged into three. The other heights, except for the orbit of 0.961(as crosses +), decrease monotonously with the increase of D (Fig. 7); the orbit of 0.471 presents a fluctuation during the decrease. Very interestingly, for the orbit of 0.961, when D is smaller than 1×10^{-6}, the amplitude tends to a constant (1/6) and when D is large, just like the other orbits, it tends to zero. However, there exists an optimized value of D at which the probability density takes its maximum value. Its height, in the interval [2×10^{-6} , 8×10^{-5}], is greater than 1/6 attained by the orbit in the absence of noise. As can be seen comparing with Fig. 3 and Fig. 5, the heights of six orbits in Fig. 7 tend to a constant (1/6) until D is equal to 1×10^{-9}, whereas for the four orbits D is 1×10^{-7}(Fig. 3) and for the eight orbits D is 1×10^{-8} (Fig. 5). This shows that the period-six sequence for $\mu = 3.846$ is more readily affected by noise than the

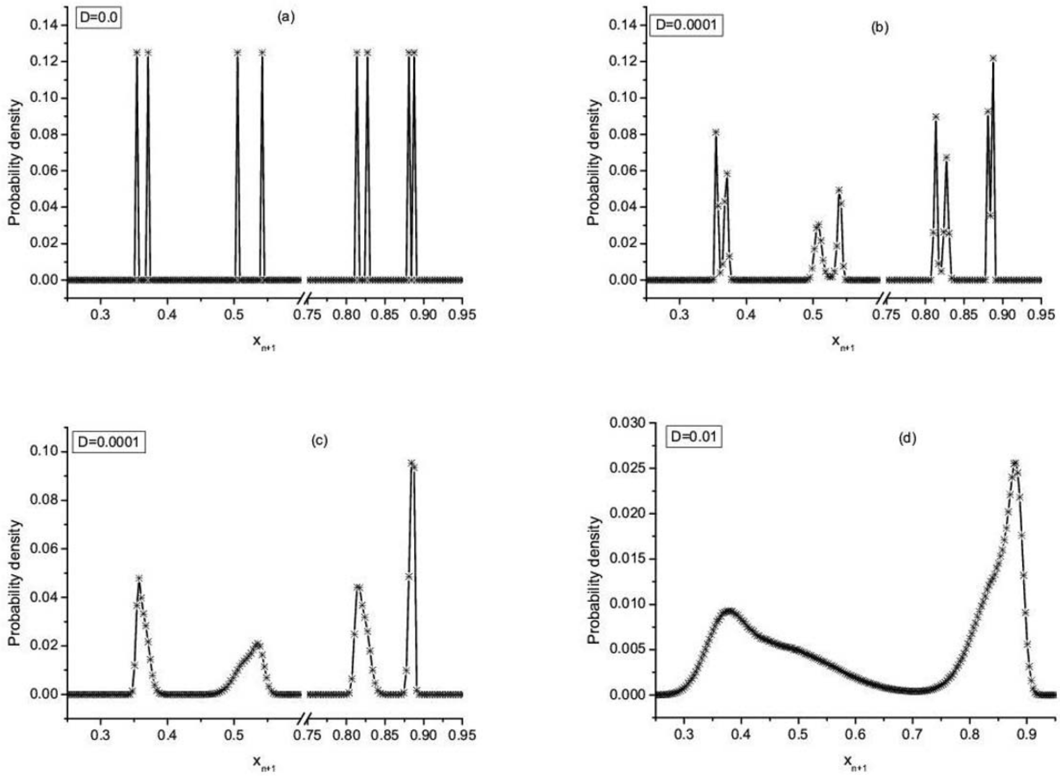

Figure 4. Plot of the normalized probability density at the bifurcation parameter value $\mu = 3.55$ for different noise intensities D: (a) $D = 0.0$, (b) $D = 0.00001$, (c) $D = 0.0001$ and (d) $D = 0.01$. 4×10^6 iterations, after an initial 6×10^6 iterations, were partitioned into 300 bins.

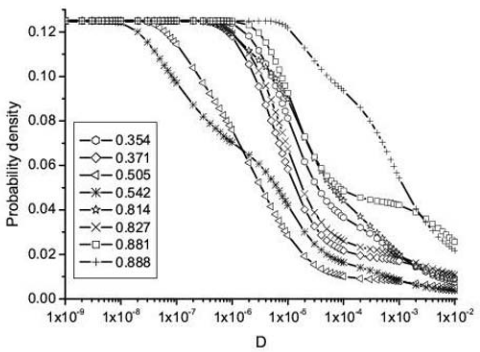

Figure 5. Plot of the normalized probability density at the bifurcation parameter value $\mu = 3.55$ for $x_{n+1} = 0.354$ (open circles ∘), 0.371 (open diamonds ⋄), 0.505 (open left triangles ◁), 0.542 (stars ∗), 0.814 (open stars ⋆), 0.827 (crosses ×), 0.881 (open squares) and 0.888 (crosses +). 4×10^6 iterations are calculated after an initial 6×10^6 iterations for each D.

period-four sequence for $\mu = 3.5$ and the period-eight sequence for $\mu = 3.55$.

4. The influence of the exponentially correlated colored noise on each periodic orbit of the four-period sequence

In order to further understand how the correlated colored noise affects the height of the probability densities for each orbit, we introduce an exponentially correlated colored noise G_n to investigate the dynamical behavior of the four-period sequence at $\mu = 3.5$. The model is written as follows:

$$x_{n+1} = \mu x_n(1 - x_n) + G_n. \tag{3}$$

Where the G_n have zero mean and correlation function: $\langle G_m G_n \rangle = \frac{D}{\tau} \exp(-\frac{|m-n|}{\tau})$ for $m, n = 1, 2, 3 \cdots$. $\langle \cdots \rangle$ denotes an ensemble average over the distribution of noise. D is the intensity of the noise and τ is the correlation

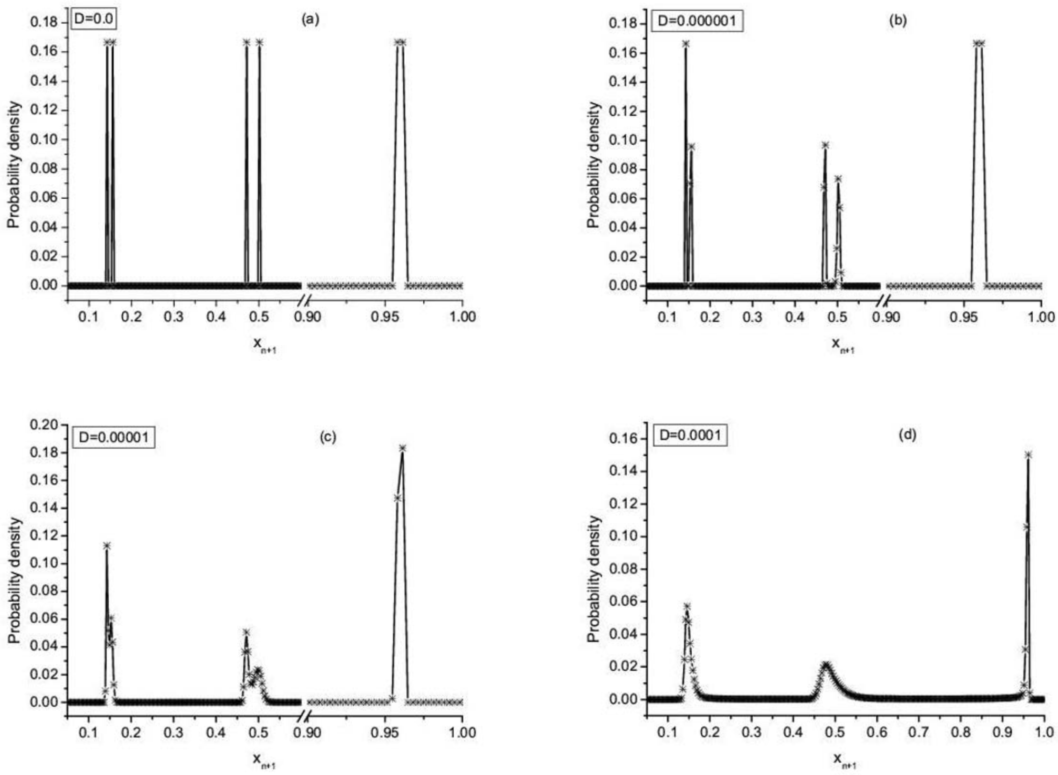

Figure 6. Plot of the normalized probability density at the bifurcation parameter value $\mu = 3.846$ for different noise intensities D: (a) $D = 0.0$, (b) $D = 0.000001$, (c) $D = 0.00001$, and (d) $D = 0.0001$. 4×10^6 iterations, after an initial 6×10^6 iterations, were partitioned into 300 bins.

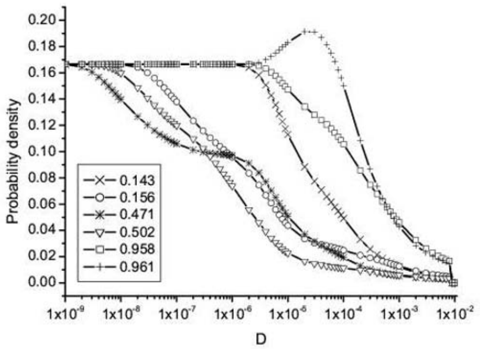

Figure 7. Plot of the normalized probability density at the bifurcation parameter value $\mu = 3.846$ for $x_{n+1} = 0.143$ (crosses ×), 0.156 (open circles ∘), 0.471 (stars ∗), 0.502 (open down triangles ▽), 0.958 (open squares) and 0.961 (crosses +). 4×10^6 iterations are calculated, after an initial 6×10^6 iterations, for each D.

time. When τ is equal to zero, G_n become a Gaussian white noise.

Fig. 8 shows the heights of the probability densities of the orbits as a function of x_{n+1} at $D = 0.01$ for τ=0.5 and 10. When τ=0, the probability density is the same as Fig. 2(d). As the correlation time τ is decreased, we also find that the period of four merges into two. Thus, the colored noise also induces the logistic map change from a four-period sequence to a two-period sequence. Fig. 9 shows the heights of the probability densities as a function of the correlation time τ for four orbits. From Fig. 9, the amplitudes of the probability densities of the four orbits increase monotonously, at different rates, when τ is greater than 0.5. The orbit of 0.827 (as up triangles △) increases quicker than the other three orbits, so this means that it is more stable. However, as we can see from Fig. 3, the orbit of 0.874 (as diamonds ⋄) in the period-four sequence is the most stable. This indicates that, under the same intensity of noise, the effects of the uncorrelated Gaussian white noise and the exponentially correlated colored noise on the period-four sequence in the logistic map are different.

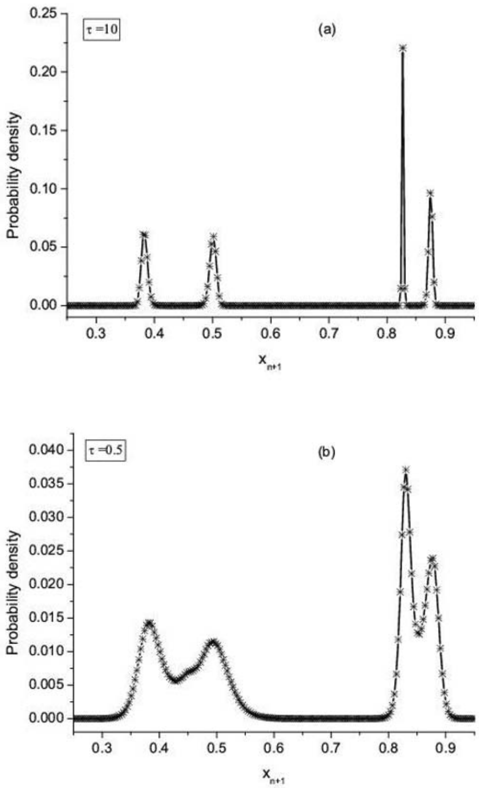

Figure 8. Plot of the normalized probability density at the bifurcation parameter value $\mu = 3.5$, for different correlation times of the colored noise: (a) $\tau = 10$, (b) $\tau = 0.5$. The intensity of the noise D is 0.01. 4×10^6 iterations, after an initial 6×10^6iterations, were partitioned into 300 bins.

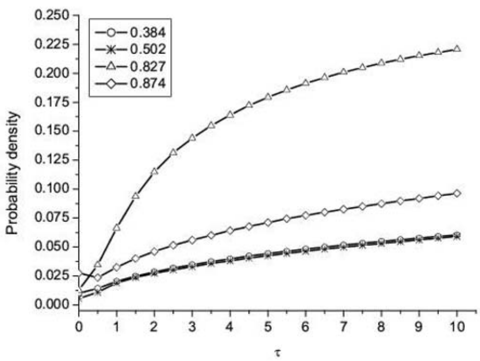

Figure 9. Plot of the normalized probability densities at the bifurcation parameter value $\mu = 3.5$ for $x_{n+1} = 0.384$ (open circles ∘), 0.502(stars ∗), 0.827(open up triangles △), and 0.874(diamonds ⋄). 4×10^6 iterations are calculated, after an initial 6×10^6 iterations, for each τ .

5. Conclusion

In this study, we investigated numerically the influence of uncorrelated Gaussian white noise and exponentially correlated colored noise on period sequences of the logistic map. We studied the probability densities of several periodic orbits that displayed merging transitions under the effects of noise. We found that the sequence for $\tau = 3.846$ is more sensitive to uncorrelated Gaussian white noise than the other two and the effects of noise on each orbit are different. However, it is unclear why the height of the probability density of the orbit of 0.961 for $\tau = 3.846$ in the interval $[\ 2 \times 10^{-6}, 8 \times 10^{-5}\ ]$ is greater than 1/6, this requires further study. We also found that, under the same intensity of noise, the effects of the exponentially correlated colored noise on the period-four sequence in the logistic map are different from the uncorrelated Gaussian white noise. These results enables us to give a detailed description of how noise influences bifurcation behavior. This paper's results should be relevant to other real dynamical systems, that undergo period-doubling bifurcations, such as fluid flows [15] and semiconductor lasers [16]. Furthermore, it is reasonable to envisage that we may be able to estimate the noise intensity by measuring changes in the probability density.

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