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Generalized Fokker-Planck equation for a class of stochastic dynamical systems driven by additive Gaussian and Poissonian fractional white noises of order α

Research Article

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Abstract: In a first stage, the paper deals with the derivation and the solution of the equation of the probability density function of a stochastic system driven simultaneously by a fractional Gaussian white noise and a fractional Poissonian white noise both of the same order. The key is the Taylor's series of fractional order $f(x+h) = E_\alpha(h^\alpha D_x^\alpha)f(x)$ where $E_\alpha()$ denotes the Mittag-Leffler function, and D_x^α is the so-called modified Riemann-Liouville fractional derivative which removes the effects of the non-zero initial value of the function under consideration. The corresponding fractional linear partial differential equation is solved by using a suitable extension of the Lagrange's technique involving an auxiliary set of fractional differential equations. As an example, one considers a half-oscillator of fractional order driven by a fractional Poissonian noise.

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1. Introduction

Fractional calculus is of increasing use in physics, and this can be explained by several factors such as the presence of internal fractional noises in the structural definition of many physical systems, the presence of fractional noises in the form of external forces, the fractal nature of time in some microscopic systems, the coarse-graining phenomenon in space and time which appear to be quite relevant in many cases, and the presence of self-similar functions involved in various problems. The common denominator of all these phenomena is fractional calculus which so appears as a framework for a unified approach..

1.1. Stochastic dynamics of fractional order in structural mechanics

Whilst Gaussian white noise and filtered Gaussian white noise provide useful efficient models for various systems

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in natural sciences involving environmental variations, a larger class of random processes including filtered Poissonian processes, appear to be much more realistic in modeling disturbances which originate from impact-type environmental variations. These representations have been used, for instance, to examine the response of bridges to moving loads [57], to model seismic loads acting on structures [9, 38], to analyse the effects of wave action on ships [50], to study the effect of political events and sudden increase of petroleum price on stock exchange dynamics [27, 28]. Grigoriu [19, 58] has estimated the efficiency of combining both Gaussian and Poissonian white noise for a wide range of problems, including earthquake, wave and traffic, effect of wind and others, and showed its efficiency as compared to using only Gaussian noises.

It appears that in a number of cases, the white noises so involved in these systems are not the standard ones (Gauss and Poisson), but rather are of fractal nature in time, and the problem which is then of interest to us, is to adapt, to modify the previous analysis methods in order to take account of this new feature.

1.2. Internal randomness in modeling dynamical systems

Many physical (and biological) systems are made up of a set of mutually interacting particles or individuals the modelling of which refers, in quite a natural way, to the Brownian motion. For instance, in biology, a useful internal randomness pattern is provided by the Wright-Fisher genetics model which is defined by the stochastic differential equation

$$dx = -\rho x(1-x)(x-\mu)dt + \sqrt{(2N)^{-1}x(1-x)}\,db(t), \quad x \in [0,1]\,, \tag{1}$$

(see for instance [10] of which the meaning can be summarized as follows. One considers a finite population (or set) of size N, with individuals (or particles) of types A and B, which generates a large number of offspring (or particles). x is the proportion of types A in the population, ρ andμ are constant selection parameters, and $b(t)$ is a Brownian motion (or Wiener-Levy process). The key is that the Brownian motion appears as the limit of the random walk defined by the selection of the two types A and B. So a problem of immediate interest is to examine what happens when one deals with a process in which the selection process chooses among more than two alternatives? Can we switch to a random walk in the complex plane [29] Furthermore, what happens when the model exhibits some memory effects? In both cases, a possible approach could be the use of fractional Brownian motion instead of the standard one.

Another example of internal randomness is provided by the diffusion of protein in cell membranes, as induced by the thermodynamical vibrations of the lipid molecules which defines the membrane itself. Assuming a spherical geometry for the cell, one is led to consider the stochastic differential equation [13]

$$d\varphi = \cot g\varphi\, dt + db, \quad \varphi \in (0,\pi)\,, \tag{2}$$

where φ is the azimuth angle which defines the coordinate of the receptors (the phenomenon is considered as being the reaction between free antigen an immune receptors on the membrane). Here, the Brownian motion is introduced analogously with the motion of particles in physics. What happens when mutual collisions between more than two particles are quite relevant?

This remark can be generalized to statistical physics. The Brownian motion is particularly relevant because usually we take into account collisions between two particles only, but if we consider a model in which collisions between more than two particles are significant, then the use of fractional Brownian motion could be quite of order.

1.3. External randomness and stability in physics

In many problems in physics one is led to analyze the equilibrium position of a system in assuming that it is subject to perturbation of random nature. The system is described by the non-random differential equation

$$\dot{x}(t) = f(x,u,t), \tag{3}$$

where $u(t)$ is an external parameter function, and close to the equilibrium position, one considers the linearized dynamics

$$\delta\dot{x}(t) = f_x(x_0,u_0,t)\delta x(t) + f_y(x_0,u_0,t)\delta u(t),$$

where the subscripts holds for the partial derivatives. Very often, it is compulsory to consider the disturbance $\delta u(t)$ in the form of a white noise because the latter is much more significant on a physical point of view. In automatic control this problem is analyzed by considering the system state itself, for instance by using Lyapunov's function, but in physics one works rather with the Fokker-Planck equation [20] which provides the expression of the probability density itself. In quite a natural way, we are led to consider the same problem when the disturbance is in the form of fractional noises.

1.4. Fractal and memory effects in physical systems

Another argument which supports introducing fractals in some physical systems, is the modeling of long-range memory effect. To shortly introduce the matter, let us consider the equations

$$dx = \alpha\rho t^{\alpha-1}dt, \quad x(0) = x_0, \quad \rho > 0, \quad 0 < \alpha < 1 \tag{4}$$

and

$$dx = \rho(dt)^{\alpha}, \tag{5}$$

which have the same solution

$$x = \rho t^{\alpha} + x_0. \tag{6}$$

The first equation is Markovian in the sense that the value of $x(t)$ depends upon the value of $x(t-dt)$ only. In contrast, in the second equation, which is equivalent to ($\Gamma(x)$ denotes the first Euler's function)

$$d^{\alpha}x = \Gamma(1+\alpha)\rho(dt)^{\alpha},$$

where $d^{\alpha}x$ is the differential of order α, $x(t)$ depends upon $x(t+(\alpha-k)dt)$, $k = 0, 1, \ldots, n$ (see the next section). Some authors refer to the memory with long-range dependence or the long-range memory effect of the second model, whilst others refer to its hereditary effects. And in quite a natural way, we are led to assume that the fractional modeling could be suitable to describe these hereditary features in some biological and physical systems. Or at least, we can try this approach to examine what it yields.
Another argument is as follows. Consider the stochastic differential equation

$$dx = f(t)dt + g(t)w(t)(dt)^{\alpha}, \quad 0 < \alpha < 1, \tag{7}$$

where $w(t)$ is a Gaussian white noise with the variance σ^2. Its moments $m_1 = \langle x \rangle$ and $m_2 = \left\langle x^2 \right\rangle$ are respectively defined by the dynamical equations

$$dm_1 = m_1 f(t)dt \tag{8}$$

and

$$dm_2 = 2m_1 f(t)dt + \sigma^2 g^2(t)(dt)^{2\alpha}. \tag{9}$$

The equation (8) is obtained by taking the mathematical expectation of (7) whilst the equation (9) is derived from the equality $dm_2 = \left\langle 2xdx + (dx)^2 \right\rangle$. So, let $m_2^{\bullet}$ denote de moment corresponding to the reference value $\alpha = 1/2$. Then the dynamics of exhibits two different behaviours depending upon whether α is lower or on the contrary larger than $1/2$. Clearly one has

$$dm_2(t) > dm_2^{\bullet}(t), \quad 0 < \alpha < 1/2$$

and

$$dm_2(t) < dm_2^{\bullet}(t), \quad 1/2 < \alpha < 1.$$

The magnitude of the deviation from the average trajectory depends upon the position of α with respect to $1/2$. In other words, α could be a useful parameter to describe the grade of variation of stochastic processes.

1.5. Fractal time in physical systems

A pervasive idea which is beginning to take place in the scientific community is that, in some physical systems, like at the quantum level for instance, we should delete the standard continuous time and assume that time is rather discontinuous, relative and random. Any physical system, at least at the microscopic level, would have its internal time τ (the term of *proper time* is already used in the relativistic physics), and would be driven by the latter, in other words the variation dx of its state variable x would be in the form

$$dx = f(x)d\tau. \tag{10}$$

Assume, for instance, that $\tau = \lambda t$, in which case it is referred to as *slow time* or *fast time* by physicists depending upon whether $\lambda < 1$ or $\lambda > 1$, then one has

$$dx = f(x)(\lambda dt). \tag{11}$$

An alternative which we suggest here would be to set

$$d\tau = (dt)^{\alpha}. \tag{12}$$

$\alpha < 1$ would define local fast time, and $\alpha > 1$ would be associated with local slow time. One could even go farther and introduce a random time τ in the form

$$d\tau = dt + \sigma w(t)(dt)^{\alpha}. \tag{13}$$

In this case, positive time would define constructive systems whilst negative time would be associated to destroying systems.

1.6. Coarse-graining space, fractal and fractional calculus

It appears that in many physical problems it would be more realistic to assume that the space in which the problem is defined, involves a coarse-graining phenomenon, what amounts to say that (loosely speaking) the points of the space in which the system is defined are not infinitely small but on the contrary have a thickness. An approach to modeling this phenomenon is to assume that the differential of space is $d\xi > dx$, and for instance, one can set $d\xi = (dx)^{\beta}$. By this way, in quite a natural way, one arrives at the framework of Hurst exponent, self-similarity, and Brownian motion of fractional order, which all of them are related to fractional calculus. Our point of view (which is one among others) is that the very reason of fractional calculus [33, 48] is to deal with non-differentiable functions and that, in this way of thought, it is expected that the fractional derivative of a constant be zero. Shortly when we are dealing with non-differentiable functions, then fractional calculus would be quite relevant.

1.7. Stochastic differential equation of fractional order

In recent years, following Mandelbrot and Van Ness [42] on the one hand, and in the wake of the above remarks on the other hand, there has been renewed interest in fractional Brownian motion, not only to model phenomena exhibiting self-similarity [6, 56] which is a form of long-range dependence, but also to improve various customary modeling involving standard Brownian motion [1], like in finance, for instance. Various approaches have been proposed to analyze fractional stochastic differential equations [11, 12, 22, 39, 60], but to the best of our knowledge, there is no sound theory for this kind of dynamics. Fractional Brownian motions are highly discontinuous, in such a manner that one has to expect to come across many problems in trying to construct a theory. So here, in order to simplify the background, instead of referring to the classical Levy-Itô's decomposition, we shall rather consider a model in the form of a non-random fractional dynamics subject to a (regular) Brownian motion.

1.8. Organization of the paper

In the following we shall derive the fractional generalized Fokker-Planck equation associated with stochastic systems driven by both a fractional Brownian motion and a fractional Poissonian noise, and we shall obtain its solution in a special case, by using an extension of the Lagrange technique for partial differential equations. The paper is organized as follows. First of all, for the convenience of the reader, in the next section we shall display a summary of fractional calculus with some new results on integral with respect to $(dt)^{\alpha}$, and then we shall outline the essential of fractional Brownian motion defined as the integral of white noise with respect to $(dt)^{\alpha}$. Then, on using probabilistic arguments, we shall carefully derive the generalized time-fractional Fokker-Planck equation. As a special case we shall obtain a rigorous derivation for the fractional Fokker-Plank equation for systems subjected to a fractional Poissonian noise only.

2. Fractional derivative revisited

2.1. Fractional derivative via fractional difference

Definition 2.1.
Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$, denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $FW(h)$ (the symbol := means that the left side by the right one)

$$FW(h)f(x) := f(x+h); \tag{14}$$

then the fractional difference of order α, $\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$, of $f(x)$ is defined by the expression

$$\begin{aligned}\Delta^{\alpha} f(x) &:= (FW-1)^{\alpha} f(x)\\ &= \sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha}{k} f\left[x+(\alpha-k)h\right],\end{aligned} \tag{15}$$

and its fractional derivative of order α (F-derivative in the following) is

$$f^{(\alpha)}(x) = \lim_{h\to 0}\frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{16}$$

This definition is close to the standard definition of derivative (calculus for beginners), and as a direct result, the α-th derivative of a constant is zero.

Proposition 2.1.
Assume that the function $f(x)$ in the Definition 2.1 is not a constant, then its fractional derivative of order α is defined by the following expression [42–44],

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)}\int_0^x (x-\xi)^{-\alpha-1} f(\xi)d\xi, \quad \alpha < 0. \tag{17}$$

For positiveα, one will set

$$f^{(\alpha)}(x) = \left(f^{(\alpha-1)}(x)\right)', \quad 0 < \alpha < 1, \tag{18}$$

$$f^{(\alpha)}(x) := \left(f^{(\alpha-n)}(x)\right)^{(n)}, \quad n \le \alpha < n+1, \quad n \ge 2. \qquad (19)$$

When $f(x)$ is a constant function, the expression (17) does not apply in the sense that it is not consistent with (16), and in order to be consistent with (16), we shall define its F-derivative as being zero.

Proof. The proof can be obtained by using Laplace transform and Z-transform and then making h tends to zero. By this way, one can show that (16) and (21) have the same Laplace transform. See for instance [13]. □

This definition will be rephrased in the following subsection.

2.2. Modified Riemann-Liouville derivative

The first definition of fractional derivative via integral which has been proposed in the literature is the so-called Riemann-Liouville definition which reads as follows

Definition 2.2 (Riemann-Liouville).
Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$, denote a continuous function. Its fractional derivative of order α is defined by the expressions (17), (18) and (19) for every $f()$ [37, 40].

But this definition gives rise to the following problem. If one makes $f(x)$ constant in (17) one finds that the α-th derivative of a constant is $x^{-\alpha}/\Gamma(1-\alpha)$ that is to say is different from zero. To circumvent this drawback, some authors [7, 8] proposed to use rather the definition

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} f'(\xi) d\xi, \quad 0 < \alpha < 1, \qquad (20)$$

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(n+1-\alpha)} \int_0^x (x-\xi)^{n-\alpha} f^{(n+1)}(\xi) d\xi, \quad n < \alpha < n+1, \qquad (21)$$

but we are reluctant to do this, for two reasons. First, according to this definition, the α-th derivative, $0 < \alpha < 1$, would be defined for differentiable functions only, whilst on the contrary, the very reason of using fractional derivative is exactly to deal with non-differentiable functions! And second, at the extreme, on including the bounds of α, this expression says that if we want to get the first derivative of a function, we must before to have its second derivative. We believe that a definition via finite difference can be of help to solve this pitfall.

2.2.1. An alternative to the Riemann-Liouville definition of fractional derivative

As a result of these remarks, we propose rather to use the following alternative to the Riemann-Liouville definition of F-derivative.

Definition 2.3 (Riemann-Liouville definition revisited).
Refer to the function of Proposition 2.1. Then its F-derivative of order α is defined by the expression

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} \left(f(x) - f(0)\right) d\xi, \quad \alpha < 0. \qquad (22)$$

For positiveα, one will set

$$f^{(\alpha)}(x) = \left(f^{(\alpha-1)}(x)\right)', \quad 0 < \alpha < 1,$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} \left(f(\xi) - f(0)\right) d\xi, \qquad (23)$$

and

$$f^{(\alpha)}(x) := \left(f^{(n)}(x)\right)^{(\alpha-n)}, \quad n \le \alpha < n+1, \quad n \ge 1. \qquad (24)$$

Let us point out that there is a strong relation between fractional derivative, self-similarity, fractal and Hurst parameter.

3. Background on Taylor's series of fractional order

3.1. Main definition

A generalized Taylor expansion of fractional order which applies to non-differentiable functions (F-Taylor series in the following) reads as follows

Proposition 3.1.
Assume that the continuous function $f : \mathbb{R} \to \mathbb{R}, x \to f(x)$ has fractional derivative of order $k\alpha$, for any positive integer k and any α, $0 < \alpha \le 1$, then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x), \quad 0 < \alpha \le 1. \qquad (25)$$

where $f^{(\alpha k)}(x) = (d^\alpha()/dx^\alpha)^k f(x)$ is the derivative of order αk of $f(x)$.

With the notation

$$\Gamma(1+\alpha k) =: (\alpha k)!,$$

the series (25) can be re-written in the form

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha \leq 1 \tag{26}$$

which looks like the classical one.

Proof.

(i) For pedagogical reasons, we first begin with a formal derivation of the usual Taylor's series. Refer to the forward shift operator $FW(h)$ defined by the relation (20). Indeed, one can write successively

$$D_h FW(h) f(x) = D_h f(x+h) = [f'(u)]_{u=x+h}$$
$$= [f'(x)]_{x \leftarrow x+h} = FW(h) D_x f(x),$$

whereby we obtain the formal identification

$$D_h FW(h) = FW(h) D_x.$$

This equality can be thought of as a formal differential equation in which $FW(h)$ is the unknown function of h to be determined, and D_x is a constant, therefore the solution

$$FW(h) = \exp\{hD_x\},$$

which is exactly the usual Taylor's series.

(ii) We can duplicate this calculus in the framework of fractional derivatives as follows. Indeed, for a discretizing span H, one has

$$D_h^{\alpha} FW(h) f(x) = \lim_{H\to 0} H^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f[x + (h + (\alpha - k)H)] = \lim_{H\to 0} H^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f[(x + (\alpha - k)H + h)]$$
$$= FW(h) D_x^{\alpha} f(x),$$

or shortly,

$$D_h^{\alpha} FW(h) f(x) = FW(h) D_x^{\alpha} f(x).$$

In other words, formally, $FW(h)$ is defined by the fractional differential equation

$$D_h^{\alpha} FW(h) = FW(h) D_x^{\alpha}, \tag{27}$$

and according to our definition of modified Riemann-Liouville derivative, its solution is

$$FW(h) = E_{\alpha}\left(h^{\alpha} D_x^{\alpha}\right), \tag{28}$$

where $E_{\alpha}(x)$ denotes the Mittag-Leffler function defined by the expression

$$E_{\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+\alpha k)}. \tag{29}$$

We then have the equality

$$f(x+h) = E_{\alpha}\left(h^{\alpha} D_x^{\alpha}\right) f(x), \tag{30}$$

therefore (25) and (26). For further details see [31–33].

□

Corollary 3.1.
Assume that $m < \alpha \leq m+1$, $m \in N - \{0\}$ and that $f(x)$ has derivatives of order k (integer), $1 \leq k \leq m$. Assume further that $f^{(m)}(x)$ has a fractional Taylor's series of order $\alpha - m =: \beta$ provided by the expression

$$f^{(m)}(x+h) = \sum_{k=0}^{\infty} \frac{h^{k(\alpha-m)}}{\Gamma[1 + k(\alpha - m)]} D^{k(\alpha-m)} f^{(m)}(x),$$
$$m < \alpha \leq m+1. \tag{31}$$

Then, integrating this series with respect to h yields

$$f(x+h)=\sum_{k=o}^{m}\frac{h^k}{k!}f^{(k)}(x)+\sum_{k=1}^{\infty}\frac{h^{(k\beta+m)}}{\Gamma(k\beta+m+1)}f^{(k\beta+m)}(x),$$
$$\beta:=\alpha-m. \tag{32}$$

In the special case when $m=1$, one has

$$f(x+h)=f(x)+hf'(x)+\sum_{k=1}^{\infty}\frac{h^{k\beta+1}}{\Gamma(k\beta+2)}f^{(k\beta+1)}(x),$$
$$\beta:=\alpha-1. \tag{33}$$

The order of the derivation in $f^{(k\beta+m)}(x)$ is of paramount importance and should be understood as $(D^{\beta})^k f^{(m)}(x)$, since we start with the fractional Taylor's series of $f^{(m)}(x)$.

3.1.1. Mc-Laurin series of fractional order

Let us make the substitution $h\leftarrow x$ and $x\leftarrow 0$ into (25), we so obtain the fractional Mc-Laurin series

$$f(x)=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+\alpha k)}f^{(\alpha k)}(0),\quad 0<\alpha\le 1. \tag{34}$$

As a direct application of the fractional Taylor's series, one has the following

Corollary 3.2.

Assume that $f(x)$, in Proposition 3.1, is α-th differentiable, then the following equalities hold, which are

$$f^{(\alpha)}(x)=\lim_{h\to 0}\frac{\Delta^{\alpha}f(x)}{h^{\alpha}}=\Gamma(1+\alpha)\lim_{h\to 0}\frac{\Delta f(x)}{h^{\alpha}},\quad 0<\alpha<1.$$

and

$$f^{(\alpha)}(x)=\Gamma\left[1+(\alpha-m)\right]\lim_{h\to 0}\frac{\Delta f^{(m)}(x)}{h^{\alpha-m}},\quad m<\alpha<m+1.$$

3.1.2. Useful relations

The equation (25) provides the useful relation

$$\Delta^{\alpha}f\cong\Gamma(1+\alpha)\,\Delta f,\quad 0<\alpha<1,$$

or in a differential form $d^{\alpha}f\cong\Gamma(1+\alpha)\,df$, between fractional difference and finite difference.

Corollary 3.3.

The following equalities hold, which are

$$D^{\alpha}x^{\gamma}=\Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)x^{\gamma-\alpha},\quad\gamma>0, \tag{35}$$

or, what amounts to the same (we set $\alpha=n+\theta$)

$$D^{n+\theta}x^{\gamma}=\Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta)x^{\gamma-n-\theta},\quad 0<\theta<1,$$

$$(u(x)v(x))^{(\alpha)}=u^{(\alpha)}(x)v(x)+u(x)v^{(\alpha)}(x), \tag{36}$$

$$(f[u(x)])^{(\alpha)}=f'_u(u)u^{(\alpha)}(x), \tag{37}$$

$$=f_u^{(\alpha)}(u)(u'_x)^{\alpha}. \tag{38}$$

$u(x)$ is non-differentiable in (36) and (37) and differentiable in (38), $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (37) and non-differentiable in (38).

Proof. *Proof of Eq. (35).*
Assume that $n<\alpha<n+1$. Then according to the equation (21) one has

$$D^{\alpha}x^{\gamma}=\frac{1}{\Gamma(1-\alpha+n)}\frac{d^{n+1}}{dx^{n+1}}\int_0^x(x-\xi)^{-\alpha+n}\xi^{\gamma}d\xi$$

$$=\frac{1}{\Gamma(1-\alpha+n)}\frac{d^{n+1}}{dx^{n+1}}\left[x^{1-\alpha+n+\gamma}\int_0^1(1-u)^{n-\alpha}u^{\gamma}du\right]$$

$$=\frac{B(1-\alpha+n,\gamma+1)}{\Gamma(1-\alpha+n)}\frac{d^{n+1}}{dx^{n+1}}x^{1-\alpha+n+\gamma}$$

therefore the equation (35), where $B(p,q)$ denotes the second Euler function. □

Proof. *Proof of Eq. (36).*
Firstly we remark that the first term of the fractional Taylor's series yields the equality

$$d^{\alpha}f(x)=\Gamma(1+\alpha)df(x)+o\left((dx)^{2\alpha}\right). \tag{39}$$

This being the case, we start with the differential

$$d(uv)=vdu+udv,$$

which, on multiplying by $\alpha!$, yields the equality

$$\alpha!d(uv)=v(\alpha!du)+u(\alpha!)dv,$$

and this is exactly

$$d^{\alpha}(uv)=vd^{\alpha}u+ud^{\alpha}v.$$

Dividing both sides by $(dx)^{\alpha}$ yields the result. □

Proof. *Proof of Eq. (37).*
One has the chain

$$\frac{d^\alpha f(u)}{dx^\alpha} = \frac{\alpha! df}{dx^\alpha} = \frac{df}{du}\frac{\alpha! du}{dx^\alpha} = \frac{df}{du}\frac{d^\alpha u}{dx^\alpha}.$$

□

Proof. *Proof of Eq. (38).*
It is sufficient to write

$$\frac{d^\alpha f(u)}{dx^\alpha} = \frac{d^\alpha f(u)}{du^\alpha}\frac{du^\alpha}{dx^\alpha}.$$

□

Corollary 3.4.
Assume that $f(x)$ and $x(t)$ are two $\mathbb{R} \to \mathbb{R}$ functions which both have derivatives of order α, $0 < \alpha < 1$, then one has the chain rule

$$f_t^{(\alpha)}(x(t)) = \Gamma(2-\alpha)x^{\alpha-1} f_x^{(\alpha)}(x) x^{(\alpha)}(t). \tag{40}$$

Proof. The α-th derivative of x provides the equality

$$d^\alpha x = \frac{1}{(1-\alpha)!} x^{1-\alpha}(dx)^\alpha. \tag{41}$$

which allows us to write

$$d^\alpha f = f_x^{(\alpha)}(dx)^\alpha$$

$$= f_x^{(\alpha)}(x)(1-\alpha)! x^{\alpha-1} d^\alpha x$$

whereby the result. □

3.2. Further results and remarks

(i) With the definition 2.3, the solution of the equation (27) is exactly the Mittag-Leffler function.

(ii) Assume that $x(t)$ is a self-similar function with the similarity index (or Hurst parameter) H, clearly

$$x(at) = a^H x(t), a > 0, 0 < H < 1. \tag{42}$$

then, one can check easily that the fractional McLaurin series of order α, $0 < \alpha < 1$, of both sides are the same.

(iii) For the sake of completeness, it is of order to point out that Osler [45] has previously proposed a generalization of Taylor's series in the complex plane, in the form

$$f(z) = \alpha \sum_{k=-\infty}^{k=+\infty} \frac{f^{(\alpha k)}(z_0)}{\Gamma(1+\alpha k)}(z-z_0)^{\alpha k}, \quad z \in C. \tag{43}$$

(vi) More recently Kolwankar and Gangal [35, 36] proved the so-called "local fractional Taylor expansion"

$$f(x+h) = \sum_{k=0}^{m} \frac{h^k}{k!} f^{(k)}(x) + \frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)} h^\alpha + R_\alpha(h),$$
$$m < \alpha < m+1, \tag{44}$$

where $R_\alpha(h)$ is a remainder, which is negligible when compared with the other terms.

Nevertheless, it is relevant and important to point out that these authors do not use the Riemann-Liouville expression of derivative as we did it, but rather define the later as the limit of a quotient involving the increment of the function on the one hand, and a so-called coarse grained mass or α-mass of a subset which is generally fractal. Loosely speaking the function is fractal because it is defined on a set which itself is fractal.

3.3. Integration with respect to *(dt)*$^\alpha$

The solution of the equation

$$dx = f(t)(dt)^\alpha, \quad t \geq 0, \quad x(0) = x_0, \tag{45}$$

is defined by the following result:

Lemma 3.1.
Let $f(t)$ denote a continuous function, then the solution of the equation (45) is defined by the equality

$$\int_0^t f(\tau)(d\tau)^\alpha = \alpha \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha \leq 1. \tag{46}$$

Proof. *Indication on the proof.*
On multiplying both sides of (45) by $\alpha!$ one obtains the equality

$$d^\alpha x = \alpha! f(t)(dt)^\alpha$$

and then it is sufficient to identify the integral in (46) with the anti-derivative of order α of $\alpha! f(t)$ to obtain the result. □

3.4. Determination of the anti derivative of order α for $1/u$

We cannot apply the formula for $D^\alpha x^\gamma$ with $\gamma = 1$, and exactly like for the pair (exponential, logarithm) we shall work by means of inverse function.
We consider the pair

$$y = f(x), x = f^{inv}(y) = g(y), \tag{47}$$

and we start with the identity

$$y = f\,(g(y))\,, \tag{48}$$

of which the α-th derivative is

$$\frac{d^\alpha y}{dy^\alpha} = \frac{\Gamma(2)}{\Gamma(2-\alpha)} y^{1-\alpha} = \frac{df}{dg} g^{(\alpha)}(y). \tag{49}$$

In the following, we shall determine which function $g(y)$ satisfies the condition

$$g^{(\alpha)}(y) = 1/y, \tag{50}$$

To this end, on combining (49) with (50) we have

$$\frac{df}{dg} = \frac{y^{2-\alpha}}{\Gamma(2-\alpha)}. \tag{51}$$

But, according to (47) one has $df = dy$, in such a manner that (51) yields

$$y^{\alpha-2} dy = \Gamma(2-\alpha) dg, \tag{52}$$

from where we obtain

$$g(y) = \frac{y^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)}. \tag{53}$$

Remark that $g(y) \to \ln y$ as $\alpha \to 1$.

4. A short background on fractional Brownian motion

4.1. Main definitions

The basic properties of the fractional Brownian motion defined as fractional derivative of Gaussian white noise can be summarized as follows [12] (See the Refs [44, 45] for the physical derivation of this process).

Definition 4.1.
Let (Ω, F, P) denote a probability space and a, $0 < a < 1$, referred to as the Hurst parameter. The stochastic process $\{b(t,a),\ t \geq 0\}$ defined on this probability space is a fractional Brownian motion $(fBm)_a$ of order a if

(i) $\Pr\{b(0,0) = 0\} = 1$;

(ii) for each $t \in \mathbb{R}_+$, $\beta(t,a)$ is an F-measurable random variable such that $E\{b(t,a)\} = 0$;

(iii) for $t, \tau \in \mathbb{R}_+$,

$$E\{b(t,a)b(\tau,a)\} = \frac{\sigma^2}{2}\left(t^{2a} + \tau^{2a} - |t-\tau|^{2a}\right), \tag{54}$$

where σ is the variance parameter.

It follows from (54) and from the Kolmogorov's continuity criterion that, for $a > 1/2$, the sample paths of $b(t,a)$ are continuous with probability one, but nowhere differentiable.

4.1.1. Further remarks and comments

(i) Unlike the semblance, the equality (54) can be very simply derived from the self-similarity property of the fractional Brownian motion.

(ii) The $(fBm)_a$ can be constructed from the classical Brownian motion $b(t) := b(t, 1/2)$ by a linear transformation of the form of an integral, and various models have been proposed [12, 42]. For further reading, see for instance [15].

(iii) If we apply formally the relation (46) above with the substitution $f(t) \leftarrow w(t)$, we have an alternative definition for $b(t.a)$, in the equation (4.5), which reads

$$b(t,a) = \left(a + \frac{1}{2}\right)^{-1} \Gamma^{-1}\left(a + \frac{1}{2}\right) \int_0^t w(\tau)(d\tau)^{a+(1/2)}. \tag{55}$$

and provides the well known equality

$$b(t) = \int_0^t w(\tau) d\tau, \tag{56}$$

when $a = 1/2$.

4.2. Maruyama notation of fractional order

The standard Brownian motion $b(t)$ of order $1/2$ and the Gaussian white noise $w(t)$ are related by the equation (56) (the Gaussian white noise is the derivative of the Brownian motion), but to some extent, the equality $db(t) = w(t)dt$ would be wrong as far as it provides the equality $E\left\{(db)^2\right\} = \sigma^2(dt)^2$. This is not an inconsistency, and is merely a result of the fact that $w(t)$ is a sequence of delta-impulses. So, to circumvent this pitfall, it is customary (at least in engineering mathematics!) to use the notation first introduced by Maruyama and which reads

$$db(t) = w(t)(dt)^{1/2}.$$

For convenience, we shall use the same contrivance here, when we shall deal with fractal noises, and we shall write

$$db(t, a) = w(t)(dt)^{a}.$$

A word of caution is of order. The equation above is only an approximation as far as, on the surface, it is not fully consistent with the correlation function (54), since it defines a process with independent increments. Nevertheless, *we ask the reader only to think of this notation as a new formal modeling, and to take a while to see the kind of results we can so expect to obtain.*

5. Generalized Fokker-Planck equation of fractional order

5.1. Stochastic dynamical equation of the considered system

5.1.1. Definition of the model

We consider the one-dimensional stochastic differential equation describing a time-dependent dynamical system subject to fractional Gaussian and Poissonian white noises, given by

$$dx = f(x,t)(dt)^{\alpha} + g(x,t)w(t)(dt)^{\alpha} + h(x,t)v_{\alpha}(t),$$
$$0 < \alpha \le 1, \qquad (57)$$

$$x(0) = x_0,$$

where x, f, g and $h \in \mathbb{R}$, $w \in \mathbb{R}$ is a Gaussian white noise with zero mean and the variance σ^2,

$$v_{\alpha}(t) = \sum_{i=1}^{N(t)} z_i \delta(t - t_i), \qquad (58)$$

where $\{z_i\}$ is a sequence of random variables with prescribed distribution. $v_{\alpha}(t)$ is a fractional scalar Poissonian white noise, that is to say a sequence of (Dirac) impulses with Poisson arrivals, characterized by the pair $(\lambda, (dt)^{\alpha})$,

$$pr(one\ arrival) = \lambda(dt)^{\alpha}, \qquad (59)$$

of the underlying fractional counting process, $N(t)$, and independent random arrivals, identically distributed with the density probability $q(z)$.
Here, in our applied mathematics framework, more or less we consider this equation on a formal standpoint, but it (the equation and not its solution!) can be soundly defined by using standard stochastic analysis [18, 24, 55].

5.1.2. On the accuracy of the modeling

On the physical standpoint, the equation (57) can be thought of as the perturbation of a non-random (deterministic) system. So, if the latter is selected in the form

$$dx = \varphi(x,t)(dt)^{\beta}, \quad 0 < \beta < 1, \qquad (60)$$

we should expect to have a general stochastic model in the form

$$dx = \varphi(x,t)(dt)^{\beta} + g(x,t)w(t)(dt)^{\alpha} + h(x,t)v_{\alpha}(t),$$
$$0 < \alpha, \beta \le 1. \qquad (61)$$

As a matter of fact, under a very simple meaningful assumption, it is possible to convert (60) in the form (57). To this end, we shall restrict ourselves to the special meaningful case $\alpha < \beta$.
Indeed, by using (35), we have the equality

$$d^{\gamma}t^{\rho} = \frac{\rho!}{(\rho-\gamma)!}t^{\rho-\gamma}(dt)^{\gamma},$$

which provides (on setting),

$$d^{\gamma}t = \frac{1}{(1-\gamma)!}t^{1-\gamma}(dt)^{\gamma} = \gamma! dt, \quad 0 < \gamma < 1.$$

We then have the equivalence

$$dt = \frac{1}{\gamma!(1-\gamma)!}t^{1-\gamma}(dt)^{\gamma}, \qquad (62)$$

and on substituting into (60), we obtain the non-random dynamics

$$dx = \frac{\varphi(x,t)}{(\gamma!(1-\gamma)!)^{\beta}}t^{\beta(1-\gamma)}(dt)^{\gamma\beta}. \qquad (63)$$

So, if we select γ such that $\gamma = \alpha/\beta$, then (62) turns to be

$$dx = \varphi(x,t)\left(\Gamma\left(1+\frac{\alpha}{\beta}\right)\Gamma\left(2-\frac{\alpha}{\beta}\right)\right)^{-\beta} t^{\beta-\alpha}(dt)^{\alpha},$$

$$= f(x,t)(dt)^{\alpha}, \quad (\alpha < \beta),$$

and we once more come across the stochastic model (57). The assumption $\alpha < \beta$ should not be considered as being restrictive, and it merely pictures the fact that the fluctuation of the random disturbance is more important than that of the non-random dynamics.

5.1.3. Long-range memory and short-range memory effect

It is well known that a stochastic system driven by a fractional Brownian motion exhibits different behaviours depending upon the value of the order α with respect to ½. It is said to have short-range memory when $0 < \alpha < 1/2$, whilst it exhibits long-range memory when $1/2 < \alpha < 1$. And of course it would be welcome that this striking property be transparent here in the above modeling. In order to clarify this point with very simple arguments, let us consider the dynaamics

$$dx = xw(t)(dt)^{\alpha}, \tag{64}$$

and its second state moment $y := \langle x^2 \rangle$. A simple calculation yields the equation

$$dy = \sigma^2 y(dt)^{2\alpha}, \tag{65}$$

of which the solution will depend upon as to whether it is with long-range memory or short-range memory.

5.1.4. Short-range memory: $0 < 2\alpha \leq 1$

Multiplying both sides of (65) by $(2\alpha)!$ and integrating provides the equation

$$d^{2\alpha} y = (2\alpha)!\sigma^2 y(dt)^{2\alpha},$$

therefore

$$\int_{y(0)}^{y} \frac{d^{2\alpha}\eta}{\eta} = (2\alpha)!\sigma^2 \int_0^t (d\tau)^{2\alpha}$$

that is to say (the subscript s holds for "short-range")

$$y_s(t) = y_s(0)E_{2\alpha}\left((2\alpha)!\sigma^2 t^{2\alpha}\right), \quad 0 < \alpha \leq 1/2. \tag{66}$$

5.1.5. Long-range memory: $1 < 2\alpha \leq 2$

Here, we re-write (64) in the form

$$(dy)^{1/2} = \pm\sigma\sqrt{y}(dt)^{\alpha}, \quad 1/2 < \alpha < 1. \tag{67}$$

By using the formula (62) which converts dt into $(dt)^{\gamma}$, one obtains

$$(dt)^{\alpha} = (\gamma!(1-\gamma)!)^{-\alpha}\, t^{\alpha(1-\gamma)}(dt)^{\alpha\gamma},$$

therefore, on setting $\alpha\gamma = 1/2$ i.e. $\gamma = 1/2\alpha$

$$(dt)^{\alpha} = \Gamma^{-1}\left(1+\frac{1}{2\alpha}\right)\Gamma^{-1}\left(2-\frac{1}{2\alpha}\right) t^{\alpha-\frac{1}{2}}(dt)^{\frac{1}{2}}. \tag{68}$$

Moreover, the fractional derivative of order ½ of $\sqrt{y}$ provides

$$(dy)^{1/2} = \Gamma^{-1}(3/2)d^{1/2}y^{1/2}. \tag{69}$$

Now, we insert (68) and (69) into (67) to obtain

$$d^{\frac{1}{2}}\sqrt{y} = \pm\sigma\,\Gamma\left(\frac{3}{2}\right)\Gamma^{-\alpha}\left(1+\frac{1}{2\alpha}\right)\Gamma^{-\alpha}\left(2-\frac{1}{2\alpha}\right)\sqrt{y}\, t^{\alpha-\frac{1}{2}}(dt)^{\frac{1}{2}} =: \pm\sigma K(\alpha)\sqrt{y}\, t^{\alpha-(1/2)}(dt)^{1/2}. \tag{70}$$

We now make the change of variable $u = \sqrt{y}$ in such a manner that (70) turns to be

$$\frac{d^{1/2}u}{u} = \pm\sigma K(\alpha)t^{\alpha-(1/2)}(dt)^{1/2},$$

therefore, on integrating,

$$u(t) = u(0)E_{1/2}\left(\pm\sigma K(\alpha)\int_0^t \tau^{\alpha-(1/2)}(d\tau)^{1/2}\right)$$

$$= u(0)E_{1/2}\left(\pm\sigma K(\alpha)\frac{\Gamma(3/2)\Gamma(\alpha+(1/2))}{\Gamma(\alpha)}\, t^{\alpha}\right)$$

$$=: u(0)E_{1/2}\left(\pm\sigma R(\alpha)t^{\alpha}\right).$$

The general solution can therefore be written in the form (here the subscript l holds for "long-range")

$$\sqrt{y_l} = C_1E_{1/2}\left(-\sigma R(\alpha)t^{\alpha}\right) + C_2E_{1/2}\left(+\sigma R(\alpha)t^{\alpha}\right),$$

$$1/2 < \alpha \leq 1, \tag{71}$$

where C_1 and C_2 are two constants to be determined by the initial conditions.

The equations (66) and (71) exhibit clearly the difference between the two dynamics.

5.2. Derivation of the fractional generalized Fokker-Planck equation

In order to circumvent some problems due to the mathematical differences between the fractional Gaussian white noise and the Poissonian white noise, we shall consider separately each one of this case, and then we shall put together the results so obtained.

5.2.1. (Step 1) Stochastic systems driven by fractional Gaussian white noise

Lemma 5.1.

The state probability density $p(x,t)$ of the stochastic process defined by the fractional differential equation

$$dx = f(x,t)(dt)^\alpha + g(x,t)w(t)(dt)^\alpha \tag{72}$$

is the solution of the fractional partial differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}p(x,t) = -\Gamma(1+\alpha)\frac{\partial}{\partial x}(fp)+\frac{\Gamma(1+\alpha)}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 g^2 p). \tag{73}$$

Proof.

The equation (72) provides the conditional expectations

$$E\{dx\,|x,t\} = f(x,t)(dt)^\alpha, \tag{74}$$

$$E\left\{(dx)^2\,|x,t\right\} = g^2(x,t)\sigma^2(dt)^\alpha, \tag{75}$$

whilst the moments of higher order can be neglected. On defining the state moment of order k by the expression

$$m_k(t) := E\left\{x^k(t)\right\} = \left\langle x^k(t)\right\rangle, \tag{76}$$

one has the equality

$$dm_k = \int_{\mathbb{R}} x^k dp = E\left\{kx^{k-1}dx + \frac{1}{2}k(k-1)x^{k-2}(dx)^2\right\}$$

which, on taking account of (74) and (75), provides

$$\int_{\mathbb{R}} x^k\left\{dp + \frac{\partial}{\partial x}(fp)(dt^\alpha) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 g^2 p)(dt)^\alpha\right\}dx = 0.$$

This equality holds for every k, and thus the coefficient of x^k must itself be zero, to yield

$$dp = -\frac{\partial}{\partial x}(fp)(dt)^\alpha + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 g^2 p\right)(dt)^\alpha. \tag{77}$$

Multiplying both sides by $\Gamma(1+\alpha)$ and dividing by $(dt)^\alpha$, we eventually obtain the F-P equation (73). □

5.2.2. (Step 2) Stochastic systems driven by Poissonian noise

Lemma 5.2.

The state probability density $p(x,t)$ of the stochastic process defined by the stochastic differential equation

$$dx = h(x,t)v(t), \tag{78}$$

is the solution of the integro-differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha}p(x,t) = -\lambda\Gamma(1+\alpha)p(x,t) + \lambda\Gamma(1+\alpha) \times \int_{\mathbb{R}} p\left[x-h(x,t)z,t\right]q(z)dz. \tag{79}$$

Proof. We re-write (78) in the more explicit form

$$x(t+dt) = x(t) + h(x,t)z.$$

Starting from this equality, a straightforward calculation yields

$$p(x,t+dt) = p(x,t)\left(1-\lambda(dt)^\alpha\right) + \lambda(dt)^\alpha\int_{\mathbb{R}} p\left[x-h(x,t)\,z,t\right]q(z)dz. \tag{80}$$

Multiplying both sides by $\Gamma(1+\alpha)$ and then dividing by $(dt)^\alpha$ we eventually obtain (79). □

5.2.3. (Step 3) Stochastic systems driven by Gausssian and Poissonian noises.

Lemma 5.3.

The probability density $p(x,t)$ of the stochastic system driven by the fractional equation (57) is provided by the generalized Fokker-Planck integro-differential equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \Gamma(1+\alpha)\left[-\lambda p - \frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 g^2 p) + \lambda\int_{\mathbb{R}} p(x-hz,t)q(z)dz\right]. \tag{81}$$

Proof. It is sufficient to combine the two results above. □

On making $\lambda = 0$ in (81) one comes across the equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \Gamma(1+\alpha)\left[-\frac{\partial}{\partial x}(fp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 g^2 p)\right]. \tag{82}$$

which is the fractional PDE associated with the process

$$dx = f(x,t)(dt)^\alpha + g(x,t)w(t)(dt)^\alpha, \quad 0<\alpha\le 1. \tag{83}$$

We have previously introduced [25] the equation (77), but without the coefficient $\Gamma(1+\alpha)$, and the same equation, but written otherwise, has been considered later by Barkai [4, 5]. Our contribution here is the presence of $\Gamma(1+\alpha)$ which allows us to switch from $dp(x,t)$ to $d^\alpha p(x,t)$.

5.3. Fractional characteristic function equation

The functional equation of the characteristic function $\Phi_X(u,t) := \langle \exp(iuX)\rangle$ can be obtained easily by using the fact that it is the Fourier's transform of the probability density function. So, it is sufficient to take the Fourier's transform of (81) to obtain the equation

$$\begin{aligned}\frac{\partial^\alpha \Phi_X(u,t)}{\partial t^\alpha} &= \alpha! iu \left\langle f(x,t)e^{iuX}\right\rangle \\ &- \frac{\alpha!}{2}u^2\left\langle \sigma^2 g^2 e^{iux}\right\rangle - \lambda\alpha!\Phi_X(u,t) \\ &+ \lambda\alpha! \int_{\mathbb{R}}\int_{\mathbb{R}} e^{iux} p\left(x-h(x,t)z,t\right) q(z)dxdz,\end{aligned} \tag{84}$$

and in the special useful case when $h()$ is a function $h(t)$ of time only, one has

$$\begin{aligned}\frac{\partial^\alpha \Phi_X(u,t)}{\partial t^\alpha} &= \alpha! iu\left\langle f(x,t)e^{iuX}\right\rangle - \frac{\alpha!}{2}u^2\left\langle\sigma^2 g^2 e^{iux}\right\rangle \\ &+ \alpha!\lambda\left(\Phi_Z\left(h(t)u\right)-1\right)\Phi_X(u,t),\end{aligned} \tag{85}$$

with the boundary conditions

$$\lim_{|u|\to 0}\Phi_X(u,t) = 0 \quad \textit{and} \quad \Phi_X(0,t) = 1, \tag{86}$$

together with initial condition derived from the boundary condition at the starting instant t_0.
The exact solution of (81) and (84) can be found for a few special systems only. One can obtain a boundary value problem for the characteristic function by using the fact that the probability density function and the characteristic function form a Fourier transform pair. In the following, we shall consider the case provided by the half-oscillator.

6. Linear half-oscillator of fractional order

6.1. Preliminary result on auxiliary system associated with FPDE

Our purpose in the following is to determine the solution of the FPDE

$$f(x,y,w)\frac{\partial^\alpha w}{\partial x^\alpha} + g(x,y,w)\frac{\partial w}{\partial y} = h(x,y,w), \quad 0<\alpha<1, \tag{87}$$

where $w : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to w(x,y)$, is subject to the initial condition

$$w(x,0) = w_0(x). \tag{88}$$

6.1.1. Fractional PDE of the first integral function

We denote by $\varphi(x,y,w)$ a first integral (function), namely $\varphi(x,y,u)$ =*constant*. We have the following result:

Lemma 6.1.
Let $\varphi(x,y,w) = const$ denote a first integral function for the FPDE (54); then it satisfies the FPDE

$$f\frac{\partial^\alpha\varphi}{\partial x^\alpha} + g\frac{\partial\varphi}{\partial y} + h\frac{\partial\varphi}{\partial w} = 0. \tag{89}$$

Proof.

(i) Let $d_x\varphi$ denote the increment of φ along x only. We then have

$$\begin{aligned} d_x\varphi &= \frac{\partial\varphi}{\partial w}d_x w + \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha\varphi}{\partial x^\alpha}(dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\frac{\partial\varphi}{\partial w}d_x^\alpha w + \frac{1}{\Gamma(1+\alpha)}\frac{\partial^\alpha\varphi}{\partial x^\alpha}(dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\left(\frac{\partial\varphi}{\partial w}\frac{\partial^\alpha w}{\partial x^\alpha} + \frac{\partial^\alpha\varphi}{\partial x^\alpha}\right),\end{aligned}$$

and on expliciting the condition $d_x\varphi = 0$, we obtain

$$\frac{\partial^\alpha w}{\partial x^\alpha} = -\frac{\varphi_x^{(\alpha)}}{\varphi'_w}. \tag{90}$$

(ii) The same (standard) calculation w.r.t. y yields

$$d_y\varphi = \left(\frac{\partial\varphi}{\partial w}\frac{\partial w}{\partial y} + \frac{\partial\varphi}{\partial y}\right)dy,$$

therefore, on equating to zero,

$$\frac{\partial w}{\partial y} = -\frac{\varphi'_y}{\varphi'_w}. \tag{91}$$

(iii) Substituting (90) and (91) into (87) yields the result (89).

□

6.1.2. On systems of fractional differential equations

Let us consider the system

$$\dot{x}(t) = f_1(x,y,t), \quad x(0) = x_0, \quad x \in \mathbb{R}, \tag{92}$$

$$y^{(\alpha)}(t) = f_2(x,y,t), \quad y(0) = y_0, \quad 0 < \alpha < 1, \tag{93}$$

where $t$denotes time, to fix the thought. We have the following result:

Lemma 6.2.
Assume that $\Psi(x,y,t) = const$ is a first integral for the system (6.6, 6.7); then one has the following FPDE

$$f_1^\alpha \frac{\partial^\alpha \psi}{\partial x^\alpha} + f_2 \frac{\partial \psi}{\partial y} + \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \frac{\partial \psi}{\partial t} = 0. \tag{94}$$

Proof.

Applying the operator $E_\alpha(dx^\alpha D_x^\alpha)\exp(dyD_y)\exp(dtD_t)$ to the function ψ , we get the increment

$$d\psi = \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \psi}{\partial x^\alpha}(dx)^\alpha + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial t} dt$$

$$d\psi = \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha \psi}{\partial x^\alpha}(dx)^\alpha + \frac{1}{\Gamma(1+\alpha)} \frac{\partial \psi}{\partial y} d^\alpha y + \frac{1}{\Gamma(1+\alpha)} \frac{\partial \psi}{\partial t} d^\alpha t$$

$$d\psi = \frac{1}{\Gamma(1+\alpha)} \left(\frac{\partial^\alpha \psi}{\partial x^\alpha} \left(\frac{dx}{dt} \right)^\alpha + \frac{\partial \psi}{\partial y} \frac{d^\alpha y}{dt^\alpha} + \frac{\partial \psi}{\partial t} \frac{d^\alpha t}{dt^\alpha} \right) (dt)^\alpha, \tag{95}$$

and we remark that

$$\frac{d^\alpha t}{dt^\alpha} = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}, \tag{96}$$

to obtain the equation (94). □

6.1.3. Auxiliary system associated with the FPDE

Lemma 6.3.
The auxiliary system of partial differential equations associated with the FPDE (87) is

$$\frac{(dx)^\alpha}{f} = \frac{d^\alpha y}{g} = \frac{(dw)^\alpha}{h}. \tag{97}$$

Derivation of these equations. The key idea is provided by the similarity between the FPDE (89) and (94). More explicitly, we re-write (89) in the form

$$\frac{1}{\Gamma(2-\alpha)} w^{1-\alpha} \left(\frac{f^\alpha}{h} \frac{\partial^\alpha \varphi}{\partial x^\alpha} + \frac{g}{h} \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial w} \right) = 0. \tag{98}$$

On comparing (98) with (94), we are led to make the substitution $t \leftarrow u$ and to set

$$f_1^\alpha(x,y,w) \equiv \frac{w^{1-\alpha} f(x,y,w)}{\Gamma(2-\alpha)h(x,y,w)},$$

$$f_2(x,y,w) \equiv \frac{w^{1-\alpha} g(x,y,w)}{\Gamma(2-\alpha)h(x,y,w)}.$$

We then have the associated differential equations (the parallels of (90) and (91))

$$\left(\frac{dx}{dw} \right)^\alpha = \frac{w^{1-\alpha} f(x,y,w)}{\Gamma(2-\alpha)h(x,y,u)}, \tag{99}$$

$$\frac{d^\alpha y}{dw^\alpha} = \frac{w^{1-\alpha} g(x,y,w)}{\Gamma(2-\alpha)h(x,y,w)}, \tag{100}$$

therefore the associated system (97).
We can now consider the linear half-oscillator of fractional order.

6.2. Basic fractional partial differential equation

In this example, we consider a half degree-of-freedom linear system modeling a simple low pass filter. The state equation for this scalar system is given by

$$dx = -ax(dt)^\alpha + v_\alpha(t)(dt)^\alpha, \quad x(0) = x_0, \quad 0 < \alpha \le 1 \tag{101}$$

where $v_\alpha(t)$ is a Poissonian white noise of order α with the parameter λ, and with the bilateral exponentially distributed pulse amplitude: clearly

$$q(z) = \frac{\beta}{2} e^{-\beta|z|}, \tag{102}$$

therefore the characteristic function is

$$\Phi_Z(u) := \int_{\mathbb{R}} e^{iux} q(z)dz = \frac{\beta^2}{\beta^2+u^2}. \quad (103)$$

The fractional generalized Fokker-Planck equation for this system is

$$\frac{\partial^\alpha}{\partial t^\alpha} p(x,t) = \frac{\partial}{\partial x}(\tilde{a}xp) - \tilde{\lambda} p(x,t) + \tilde{\lambda} \int_{\mathbb{R}} p(x-z,t)q(z)dz, \quad (104)$$

with

$$\tilde{a} := \Gamma(1+\alpha)\,a, \quad \tilde{\lambda} := \Gamma(1+\alpha)\,\lambda \quad (105)$$

On taking the Fourier transform of this equation, we obtain the fractional partial differential equation which provides the characteristic function $\Phi(u,t)$ associated with the unknown $p(x,t)$, clearly

$$\frac{\partial^\alpha}{\partial t^\alpha}\Phi(u,t) - \tilde{a}\,u\frac{\partial}{\partial u}\Phi(u,t) + \frac{\tilde{\lambda}u^2}{\beta^2+u^2}\Phi(u,t) = 0,$$

$$0 < \alpha \leq 1, \quad (106)$$

with the boundary condition $\Phi(0,t) = 1$ and the initial condition $\Phi(u,0) = 1$.

6.2.1. Solution in the special case when $\alpha = 1$

Let $\Phi_1(u,t)$ denote the characteristic function associated with $\alpha = 1$; using the Lagrange technique, we obtain the auxiliary system

$$\frac{dt}{1} = -\frac{du}{\tilde{a}u} = -\frac{\left(\beta^2+u^2\right)}{\tilde{\lambda}u^2}\frac{d\Phi_1}{\Phi_1}, \quad (107)$$

therefore the equations

$$du = -\tilde{a}u\,dt, \quad (108)$$

and

$$\frac{\tilde{\lambda}}{\tilde{a}}\frac{u^2du}{\beta^2+u^2} = u\,\frac{d\Phi_1}{\Phi_1}. \quad (109)$$

We then obtain the general solution

$$\Phi_1(u,t) = \left(\beta^2+u^2\right)^{\lambda/2\tilde{a}} F\left(ue^{\tilde{a}t}\right), \quad (110)$$

and the initial and boundary conditions provide

$$\Phi_1(u,t) = \frac{\left(\beta^2+u^2\right)^{\lambda/2\tilde{a}}}{(\beta^2+u^2e^{2\tilde{a}t})^{\lambda/2\tilde{a}}}. \quad (111)$$

6.3. Solution of the basic fractional partial differential equation

The auxiliary system associated with (106) is

$$\frac{(dt)^\alpha}{1} = -\frac{d^\alpha u}{\tilde{a}u} = -\frac{\beta^2+u^2}{\tilde{\lambda}u^2}\frac{(d\Phi)^\alpha}{\Phi} \quad (112)$$

therefore the two equations

$$u^{(\alpha)}(t) = -au(t), \quad (113)$$

$$\frac{(d\Phi)^\alpha}{\Phi} = \frac{\lambda}{\tilde{a}}\frac{u\,d^\alpha u}{\beta^2+u^2}. \quad (114)$$

(i) As usual now, the solution of (113) is $u = u_0E_\alpha(-at^\alpha)$ and provides the first integral

$$uE_\alpha^{-1}(-\tilde{a}t^\alpha) = const \quad (115)$$

(ii) The solution of the equation (114) is based on the following two results

First: one has the equality (see the equation (37))

$$(f\,[g(x)])^{(\alpha)} = f'_g(g)g^{(\alpha)}(x). \quad (116)$$

as a result of the property

$$\Gamma(1+\alpha)df = \frac{df}{dg}\left(\Gamma(1+\alpha)dg\right).$$

Second: one has the inference (see Subsection 3.4)

$$\text{If} \quad g^{(\alpha)}(\Phi) = \frac{1}{\Phi} \quad \text{then} \quad g(u) = \frac{\Phi^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)} \quad (117)$$

or again

$$\int\frac{(d\Phi)^\alpha}{\Phi} = \frac{\Phi^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)}. \quad (118)$$

By using these results, we integrate both sides of (114) to obtain (Kdenotes a constant)

$$\ln\frac{\left(\beta^2+u^2\right)^{\lambda/2a}}{K} = \frac{\Phi^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)}, \quad (119)$$

therefore the first integral (function)

$$\left(\beta^2+u^2\right)^{\lambda/2a}\exp\left(-\frac{\Phi^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha}\right) = const. \quad (120)$$

(iii) On combining (115) and (120), the general solution of the equation (106) can then be written in the form

$$\left(\beta^2+u^2\right)^{\lambda/2a}\exp\left(-\frac{\Phi^{\alpha-1}}{(\alpha-1)\Gamma(2-\alpha)}\right)=F\left(uE_\alpha^{-1}(at^\alpha)\right). \tag{121}$$

The explicit expression of $F(x)$ is given by the boundary condition $\Phi(0,t) = \Phi(u,0)=1$, and we have eventually

$$\frac{\Phi^{\alpha-1}-1}{(\alpha-1)\Gamma(2-\alpha)}=\ln\left(\frac{\beta^2+u^2}{\beta^2+u^2E_\alpha^{-2}(at^\alpha)}\right)^{\lambda/2a}. \tag{122}$$

Remark that when $\alpha = 1$, then $u^{\alpha-1}\cong 1+(\alpha-1)\ln u$, and one finds exactly the solution $\Phi_1(u,t)$ in (111).

7. Further remarks and comments

7.1. Recalling of main results

Starting from the prerequisite that it would be suitable that the fractional derivative of a constant be zero, we have slightly modified the Riemann-Liouville definition and we so arrived at a fractional Taylor's series which provides fractional differential relations which could serve as a point of departure for a new fractional calculus in which, loosely speaking, the exponential function is replaced by the Mittag-Leffler function.

This fractional calculus allowed us to soundly derive a class of Fokker-Planck equations which are fractional in time, and we did it by using a calculus quite parallel to the classical probabilistic one. In this way, we think that it could complete some other works in which these equations are not derived by a chain of probabilities rules, but rather are introduced more or less formally [2, 3, 23, 34, 41, 46, 51, 53]. In addition, our probability density of fractional order appears to be a candidate to fully support these equations on the theoretical standpoint.

Up to now the techniques which are generally used for solving fractional PDE are the separation of variables, Laplace's transform and Fourier's transform [2, 5, 16, 17, 21, 54, 59] which converts the FPDE into fractional differential equations. Here, we have suggested an alternative which provides a new viewpoint based on the theory of first integral in the theory of partial differential equations. .

7.2. Suggestions for further research

A topic of investigation for the future is to examine whether it is possible to extend the Lagrange method for solving fractional PDE to the case when the FPDE involves several partial derivatives of different order.

Fractal begins to appear in the study of population dynamics [14, 19], and of course it could be of interest to see what kind of results one could obtain with the fractional calculus above.

When we derived Fokker-Planck equations fractal in space and time, we came across the concept of probability density of fractional order which we introduced previously, and their remains to deepen this relation, and to get new results, mainly with regard to quantum mechanics.

Another topic of interest would be to examine in which way one can apply this formalism to study dynamical systems which are subject to Levy's noise [52] on the one hand, and are described by probability density functions of fractional order [32] on the other hand.

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